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ECONOMETRICA

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THE NOTION OF INVOLUNTARY ECONOMIC DECISIONS[1]

BY TRYGVE HAAVELMO

The Keynesian definition of involuntary unemployment has given rise to much controversy. According to this definition it is useless to discuss problems of involuntary unemployment within a consistent economic model if one of its equations is the classical supply function of labor, because there can be no involuntary unemployment within such a model Similar problems arise whenever one tries to demonstrate the possibility of involuntary individual economic decisions *within a given model* of economic behavior It is the contention of the present writer that the notion of involuntary economic decisions, to become meaningful, must be derived from a comparison *between alternative economic models*, or frameworks, under which society may collectively choose to operate.

1. INTRODUCTION

IN SPITE of all modern writings on "the Age of Plenty" and "the Problems of Effective Demand" the basic problems of economic science remain the same. They are the problems that have their origin in the universal and ever-present shortage of goods and services in relation to human wants. For the same reason it is hardly ever possible to speak of voluntary economic action or decision in any absolute sense; for if there were no obstacles to economic choice, we should certainly experience that economic decisions generally would be different from what they are, or have been, in any society. In an absolute sense, therefore, most economic decisions could properly be classified as "involuntary." If this were what the economists are thinking of when they use the phrase "involuntary," then not much more could be said on the subject. However, it is obvious that the phrase "involuntary" action, or decision, etc., when used in economic theory, is meant to cover concepts that are more specific, and less trivial.

Here is not the proper place to enter into a lengthy philosophical or philological discussion of the types of human action that should rightly be called "involuntary." But, as economists, we do want to know rather precisely what we mean when we talk about "involuntary unemployment," "involuntary saving," or the like. Such notions have played—and are playing—an important part in economic discussions, discussions which probe deeply into the substance of economic theory. Quarreling

[1] This article will be included in Cowles Commission Paper, New Series, No. 38.

are the changing attitudes towards the question of social justice, individual freedom, etc. And, finally, there is the problem of possible changes in taste and behavior of the individual groups and, therefore, changes in the market point x chosen under a given system S and given technological constraints.

The notion of involuntary economic decisions as developed above, vague and complicated though it may be, does help to bring out some of the main problems involved in decisions upon economic policy. Consider for example a simple statement like this: "Production is too low compared with the amount that people are willing to produce." If this is to be more than a loose phrase, we have to ask why it is that such a situation can prevail. In general, it is of little use to argue that the actions of the individual groups are irrational or erroneous. Instead we shall have to pose the following question. The present situation being (S', x'), is it possible to propose another feasible and practical system S'' such that x'' involves more production than x' and such that a collective decision would be in favor of S''?

These ideas do not, of course, imply that economists and other experts should study or propose only systems that *today* have a good political chance of being adopted. Quite the contrary. The broader the selection of practical, feasible systems and the more widespread the knowledge about the working of such systems, the better, presumably, is the chance of obtaining wise decisions.

To test the use of the definition of "involuntariness" as developed above, let us compare it with the definition of "involuntary unemployment" as given by Lord Keynes. He describes a state of involuntary unemployment as a situation in which more labor would be forthcoming at the prevailing wage than the amount of labor actually employed.[2] It would seem natural, according to this description, to define the amount of involuntary unemployment as the difference between the amount of labor that would be offered at existing wages and the amount actually employed. But then one has the following problem: If (S', x') is the existing economic situation, is it possible to devise a feasible and acceptable system S'', leading to a new situation (S'', x'') in which x'' involves the same wage rate as x', but where x'' differs from x' with regard to employment by an amount equal to the amount of "involuntary unemployment" implied in the Keynesian definition? Clearly it is trivial to compare the prevailing situation with something that might not be logically possible or practically feasible or collectively acceptable. That would be as trivial as to say that people are "involuntarily" consuming less than they want because there are not enough goods and services for everybody.

[2] J. M. Keynes, *General Theory of Employment, Interest and Money*, New York: Harcourt, Brace and Co., 1936, pp. 10–15.

This is not saying that the possibilities of policy changes that could improve upon economic welfare are more limited than visualized by the majority of economists. On the contrary, the possible gains by collective policy decisions and by providing economic knowledge as a basis for such decisions are probably underestimated. But we need to be clear on the point that all alternatives worth considering must represent consistent and workable systems and that in addition there is a question of choosing between them, which choice is not decided solely by consideration of the location of a market point x. The choice also depends on S.

3. THE IDEA OF "OVERDETERMINED" ECONOMIC SYSTEMS[3]

The notion of involuntary economic decisions, of which the Keynesian definition of involuntary unemployment is an example, is—stated generally—that a market point x' entails involuntary action to the extent that it differs from another "more desirable" market point x''.

We have argued that this is probably not a very fruitful approach, unless the point of comparison, x'', be derived from a workable and acceptable system S''. And this requirement would take us back to the notion of involuntary action which we have suggested in the foregoing, and according to which the "involuntariness" consists in maintaining S' when S'' would be collectively preferred.

Even more complex is the idea of "overdetermination" of the "voluntary" market point. The idea seems to be as follows. Consider the individual groups acting under given technological constraints and a given system, S, of organizational rules or constraints. Suppose that each of the groups thinks that a certain mode of action is "natural" or "desirable" under the conditions given. It may be possible to describe such a complex of modes of action by a system of simultaneous equations between the variables X, incorporating the technological and organizational constraints as interpreted by the various individual groups. We may easily conceive of such a system of equations as being logically contradictory, or "overdetermined." This means that there exists *no* market point x satisfying the "desired" system of equations. In that case the individual groups obviously have to do *something else.* Thus, from the knowledge of such a logically contradictory system we can say something about what the individual groups can *not* do. But obviously this does not in general determine what they *will* choose as the alternative. All we can say is this: The market point x must by its very nature be a unique observable point at any time. *Therefore, if we choose to say*

[3] See, for example, Ragnar Frisch, "Overdeterminateness and Optimum Equilibrium," *Nordisk Tidskrift for Teknisk Økonomi*, No. 37 (1–4), 1948, pp. 95–105, and Don Patinkin, "Involuntary Unemployment and the Keynesian Supply Function," *The Economic Journal*, Vol. 59, September, 1949, pp. 360–383.

that the market point x is determined by a system of simultaneous equations, this system must permit the solution x.

One could perhaps conceive of an "overdetermined" system as a link in a virtual process of mental trials and errors by which the individual groups approach the problem of *finding a workable mode of action.* This mental experiment is in fact exactly similar to the one that we sometimes use in order to understand how the individual groups, by mutual trials and errors, find the *form* of a system of behavior relations that actually does permit of a solution. It does not seem very fruitful—at least not from the point of view of economic policy—to define the change-over from a set of mutually contradictory modes of action to one that is not contradictory as "involuntary," for clearly there is no collective decision that could fully gratify the wishful thinking that is involved in a contradictory system of economic relations.

A few more comments may be added regarding the possible operational meaning of the processes of trial and error that were mentioned above. When we consider such processes, we usually think of them as a *chain*, one trial suggesting what the next should be. This being the case, one could think of a much wider set of rules of behavior, always consistent, and describing the following process. First, a virtual process of trials and errors running through systems of behavior equations and converging infinitely fast and uniquely upon one having a solution x permissible under the prevailing technological constraints and the existing organizational system; next, a reconsideration, by each group individually, whether, from its individual point of view, it has actually chosen a mode of action which is in the group's own interest; then a new process of choice of behavior patterns converging upon some other system of relations that has a solution, and so forth. In this way it may be that, under a given system S', a market point x' is established such that all the groups think they are acting in their own best interest. Still the market point x' may be very bad from the point of view of economic welfare compared to what might be obtainable under some alternative system S''. That is to say, the system S' is involuntarily maintained because collective action is lacking.

These are the cumbersome lines of thought that one has to go through if one wants to visualize how "market equilibrium" is established in the field of *static* theory. Such reasoning may perhaps serve as a means of abstract interpretation of a market mechanism, but it is hardly a realistic description of the processes that actually take place. It represents no operational method of decision from the point of view of the acting economic groups. For a more realistic description of the modes of action under a certain economic system it is necessary to turn to the field of *dynamics*. A dynamic interpretation would be as follows.

Consider a given organizational system S' and given technological constraints. Let $x(t)$ be any market point at time t, subject only to the organizational and technological constraints. Given this market point, the various individual groups may desire to take certain actions which are open to them to change the market point. How is it possible at the same time to accept a given market point $x(t)$ and to desire to change it? In the static theory it seems hopeless to answer this question. But in a dynamic theory the answer is simple. In a dynamic theory the parameters of action of the individual groups need not be the components of the market point itself, but, for example, the rates of change, of various orders, of these components. If the rates of change that represent parameters of action of the individual groups are zero, this fact may be taken to mean that, within the alternatives of choice that are open to each group acting individually under the given technological and organizational constraints, there is no desire on the part of the individual groups to change the market point or the manner in which the market point is developing. However, even if the market point $x(t)$ has this property for all values of t, it does not necessarily mean that the system S' is "optimal" in the sense of economic welfare. But changing the system S' requires *collective* action, which might not be forthcoming for reasons we have already explained.

The system S' and the technological constraints may be such that there is *no admissible market point* or market development, $x(t)$, for which all the rates of change that represent parameters of individual action *are zero* identically in t. In fact, the behavior of the individual groups acting—from their point of view—in their own best interest may lead to a development of the market point through time which only increases the individual desires and efforts to change it. Under such a process the market point itself may gradually become either more preferable or less preferable, for the efforts to change the market point will depend both on the market point itself and on the apparent further opportunities of changing it. Under such a development, the market point being in motion, it may be more difficult to get clear views on the eventual need for collective decisions to change the organizational system, unless the development goes steadily in such a bad direction that people become "fed up."

4. THE PROBLEM OF OPTIMAL ORGANIZATIONAL SYSTEMS

We have seen that the technological conditions prevailing in a given situation and the organizational system under which the various groups must operate represent constraints upon the freedom of choice of the individual groups in their attempt to influence the market point x in the direction of their own best interest. Would it then not seem obvious that

the optimal organizational system would be one under which the individual groups could have all the freedom of action that the technological constraints permit? Or expressed in different words, are not the result of collectively chosen organizational constraints in general a loss in "average economic welfare," a loss found necessary merely in order to ensure certain standards of equalization and social justice? The weaknesses of such a conclusion are not difficult to detect. The conclusion is based upon the implicit assumption that the individual groups, given certain technological constraints and a collectively agreed upon organizational system, would somehow get together, or be led by Adam Smith's "invisible hand," to survey the opportunities of choice in order to make some sort of optimal decision. But this means in fact some sort of collective action, and it is precisely this sort of action which is external to the sphere of reasoning of the individual group as such.

A particular group acting under the prescribed constraints of a general organizational system is also faced with another set of constraints, viz. those involved in the known, or assumed, individual *response of the other groups*. These additional constraints, together with the technological and the collectively chosen organizational constraints, lead to the individual behavior patterns that determine the market point x and thus exhaust all the original degrees of freedom of the market point. In the final analysis it is, therefore, not a question of whether or not these are constraints upon the market point but rather a question of how these constraints have come about. To argue that the constraints that have, so to speak, "made themselves" in the market are optimal would certainly seem rather arbitrary. One need only think of such alternatives as that of having a police force to keep law and order compared to that of a more free and "natural" system of individual methods of protection.

Finding an optimal organizational system is, however, not a question of choosing collectively a system which at a given time could give the "best possible" market point. Nor is it a question of choosing an organizational system that seems the most "natural" or reasonable from the point of view of behavior of the individual groups. The choice must be based upon a survey of various feasible *economic situations* (S, x). The choice will be influenced *both* by S *and* by x, and in a manner that cannot be expected to remain invariant over time.

University Institute of Economics, Oslo

THE DETERMINACY OF ABSOLUTE PRICES IN CLASSICAL ECONOMIC THEORY[1]

By W. Braddock Hickman

This paper takes issue with Don Patinkin, who recently argued that the simultaneous equations of classical economic theory are necessarily inconsistent, and that the classical attempt to determine real prices in the real sector of the economy and absolute prices in the monetary sector involves logical contradictions. Since a large part of his argument hinges on a misunderstanding of just what it was that the classical school assumed, the present paper restates the classical theory so as to emphasize its postulational bases A simple example is then employed to demonstrate that it is quite possible to set up a consistent classical system in which relative prices are determined in the real sector independently of absolute prices in the monetary sector. Next the details of Patinkin's analysis are examined for the flaws that led him to believe that such a system could not be set up. The paper closes with some observations on the ways in which determinacy can be built into a system and on the generally unsatisfactory state of equilibrium theory.

SUMMARY

In a recent article in this journal, Don Patinkin argues that classical economic systems such as those developed by Cassel are inconsistent.[2] In particular, he attacks the classical dichotomy which stipulates that relative prices are determined in the real sector of the economy and absolute prices in the monetary sector.

In his attack Patinkin criticizes neither the basic assumptions of the theory nor the theoretical conclusions from the standpoint of their correspondence with the known facts of economic life. He confines himself solely to the purely formal problem of internal mathematical consistency. In this paper we, too, discuss only the logical aspects of the problem, without regard to matters of realism.

Economists in dealing with this question have usually assumed as a principle that a set of demand and supply equations have a solution when the number of equations equals the number of unknowns. The principle is a nonrigorous extension of the theory of linear equations. A *nonhomogeneous* linear system *may* have a unique solution when there are n equations and n unknowns. A *homogeneous* linear system, on the other hand, *may* have a unique solution when there are $n - 1$ equations in n unknowns, since the n unknowns reduce to only $n - 1$ ratios. This

[1] I am greatly indebted to David Durand and Lawrence R. Klein of the National Bureau of Economic Research for their helpful comments on an earlier draft of this paper.

[2] Don Patinkin, "The Indeterminacy of Absolute Prices in Classical Economic Theory," Econometrica, Vol. 17, January, 1949, pp. 1–27.

would imply that the economist who counts equations and unknowns should pay particular attention to the matter of homogeneity. Most of Patinkin's argument revolves around this point.

The Casselian system, in its real part, postulates $n - 1$ commodity equations specifying equality of demand and supply. Since the unknowns in these equations are the $n - 1$ prices, we have the same number of equations as unknowns. It is usually assumed, however, that the system is homogeneous, which means that the $n - 1$ prices reduce to $n - 2$ price ratios, or relatives. Patinkin argues from this that a solution can be obtained from only $n - 2$ of the equations and that the classical system is overdetermined. He concludes that relative prices cannot be determined in the real sector of the economy.

Since the solution of the real segment of the classical system, if possible, provides only price-relatives, the classical economists postulated a constraint in the monetary sector to determine absolute prices. An example is the Cambridge equation, which relates the demand for money to the quantity of money via an institutionally determined constant. Patinkin denies that such an equation will provide the necessary solution, again basing his argument on the homogeneity of the classical demand function for money.

It is quite easy, however, to set up consistent classical systems, as we demonstrate by a simple model in which relative prices are uniquely determined in the real sector and absolute prices in the monetary sector. The weak point in the deductive chain, which has led Patinkin and others astray, is the principle of counting equations and unknowns to insure a solution. In general, the consistency of a system cannot be determined in such a convenient way. Safety demands that each specific set of equations be examined to determine the specific consistency of the system. In the treatment of classical models, however, too little is known about the nature of the functions (their shape and the magnitude of their parameters) to permit a specific test. We can neither prove by the principle of enumeration nor by specific testing that classical systems are inconsistent. And, unfortunately, the argument cuts two ways: we cannot prove without information external to the theory that such systems are consistent.

This negative conclusion will be unwelcome to some friends of equilibrium theory who will feel cut off from a complacent acceptance of the consistency of economic systems. The way out of the difficulty lies, we believe, in a more careful statement of the axiomatic bases of theory. We are at liberty to assume a priori that it is possible to express economic relationships by means of consistent equations, or we can assume that this is not possible. Patinkin assumes at one point, for example, that the classical equations are independent, within a restricted universe of equa-

tions. This is tantamount to an a priori assumption of inconsistency. Thus Patinkin disposes of the classical system without giving it a test. And in the same way we could dispose of any general system. Actually, however, the classical economists selected the alternative hypothesis: they assumed consistency rather than independence. This is the more constructive approach since it may lead to operational (statistically varifiable) results.

I. THE STRUCTURE OF CLASSICAL SYSTEMS

The structure of classical systems may be exhibited symbolically as follows:[3]

$$\text{(a.1)} \qquad D_i \equiv f_i(p_1, \cdots, p_{n-1}), \qquad \text{(6.1)}$$

$$\text{(a.2)} \qquad S_i \equiv g_i(p_1, \cdots, p_{n-1}), \qquad \text{(6.2)}$$

$$\text{(a.3)} \qquad D_i = S_i, \qquad \text{(6.3)}$$

$(i = 1, \cdots, n)$, where the first $n - 1$ equations in each set refer to commodities (that is, to goods and services) and the last refers to money.[4] The market demand and supply functions are defined in (a.1) and (a.2) in terms of all $n - 1$ of the prices. (The price of money is unity and therefore does not enter these functions.) The equations (a.3) express the fact that in equilibrium all of the markets are cleared.

It is sometimes convenient to consolidate the demand and supply functions (a.1)–(a.2) into excess demand functions and to express the equilibrium conditions (a.3) in terms of them as follows:

$$\text{(a.4)} \qquad X_i(p) \equiv D_i(p) - S_i(p),$$

$$\text{(a.5)} \qquad X_i(p) = 0,$$

$(i = 1, \cdots, n)$, where (p) is written as a shorthand expression for the price-vector $(p_1, \cdots, p_{n-1})$.

1. *The first basic assumption of classical systems is that the demand and supply functions for commodities are unaffected by changes in the price level,* that is,

$$\text{(b.1)} \qquad D_i \equiv f_i(p) \equiv f_i(\lambda p) \equiv \lambda^0 f_i(p),$$

$$\text{(b.2)} \qquad S_i \equiv g_i(p) \equiv g_i(\lambda p) \equiv \lambda^0 g_i(p),$$

$$\text{(b.3)} \qquad X_i \equiv f_i(p) - g_i(p) \equiv \lambda^0 X_i(p),$$

[3] See, for example, Gustav Cassel, *The Theory of Social Economy* (translation), New York: Harcourt, Brace and Co., 1932, pp. 138–155. Cassel's discussion starts with the theory of markets rather than with the theory of individual behavior, which logically precedes it. Walras and Pareto, however, built up similar systems from first principles, i. e., from the theory of individual behavior.

[4] The right-hand numbering system is Patinkin's, the left-hand is that of the present paper.

($i = 1, \cdots, n - 1$), which is another way of saying that the D_i, S_i, and X_i are homogeneous functions of degree 0, identically in the p's.

Since the effective demand for commodities equals the supply of money (a similar identity holding between supply of commodities and demand for money), we have

$$D_n \equiv f_n(p) \equiv p \cdot g^*, \tag{b.4}$$

$$S_n \equiv g_n(p) \equiv p \cdot f^*, \tag{b.5}$$

and hence

$$X_n \equiv f_n(p) - g_n(p) \equiv p \cdot (g^* - f^*), \tag{b.6}$$

where (p) is the vector of prices and (f^*) and (g^*) are vectors of the demand and supply functions of commodities. Since (f^*) and (g^*) are homogeneous of degree 0, the functions (b.4)–(b.6) are homogeneous of degree 1, that is,

(b.7) $$f_n(\lambda p) \equiv \lambda^1 f_n(p), \qquad (10.5)$$

(b.8) $$g_n(\lambda p) \equiv \lambda^1 g_n(p),$$

(b.9) $$X_n(\lambda p) \equiv \lambda^1 X_n(p),$$

identically in λ and the p's.

When (b.6) is written in indicial notation, we have

(c.1) $$X_n \equiv (D_n - S_n) \equiv -\sum_{i=1}^{n-1} p_i X_i, \qquad (6.6)$$

which is Walras' Law. It states that the excess-demand function for money necessarily vanishes with the excess-demand functions for commodities. The equations (a.5) are therefore dependent.[5] Two situations are now possible. Either (1) the $n - 1$ commodity equations in (a.5) are inconsistent and the system has no equilibrium position, or (2) the commodity equations in (a.5) are consistent and the monetary equation is redundant as an equilibrium equation.

2. Without further information about the parameters of the system or about certain side conditions such as Say's Law (to which we shall refer later), there is no way of knowing a priori whether or not (a.5) is consistent. *Therefore, the second basic assumption needed for the construction of classical systems is that the commodity equations are consistent.*

It is important to see that the second basic assumption is not redundant, and that without it the equations may be inconsistent. Consider, for example, the system (a.5) and the conditions (b.3), that is,

$$X_i(p) \equiv X_i(\lambda p) = 0, \qquad (i = 1, \cdots, n-1).$$

[5] A set of n equations is dependent if a solution of $n - 1$ of them is necessarily a solution of the other.

Setting $\lambda = 1/p_{n-1}$, $p_{n-1} \neq 0$, in this system yields

$$\text{(d.1)} \quad X_i = X_i(p_1/p_{n-1}, \cdots, p_{n-2}/p_{n-1}, 1) = 0, \quad (i = 1, \cdots, n-1),$$

which indicates that the $n-1$ commodity equations are expressible in terms of $n-2$ price ratios. Because there are more equations than price-ratios, such equations may be overdetermined and inconsistent; but this is not necessarily the case. The classical school assumed consistency, this assumption being made explicitly by Cassel[6] and Divisia and implicitly by Pareto and Walras. In fact, the consistency of (d.1) is precisely what the classical school meant by the assertion that relative prices are determined in the real sector of the economy.

A third assumption is needed if absolute prices are to be determined in a monetary economy. This is evident if we consider the set of $n-2$ price ratios which is the solution of (d.1). Since any set of $n-1$ absolute prices reducible to this set of ratios is a solution of (d.1), the price level is indeterminate in the real sector of the economy.[7]

3. This indeterminacy is removed in the classical theory by *the third basic assumption that a constraint in the monetary sector determines the absolute level of prices.* Several approaches are open to us in constructing an appropriate classical constraint, but most of them involve the introduction of complicating variables, such as interest rates, security prices, etc. We illustrate the method in the simplest possible way by assuming, with Cassel, that (D_n), the demand for money considered as a flow, adjusts itself via the price level to (M), the quantity of money considered as a stock; thus:

$$\text{(e.1)} \qquad M = KD_n(p) = K(1/\lambda)D_n(\lambda p), \qquad (10.4)$$

where K is an institutionally determined constant (the Cambridge "K"). Since $D_n(\lambda p)$ is known once the relative prices are known, (e.1) fixes λ and the price level is determined.[8]

To summarize, three basic assumptions are made in the construction of a classical system: (1) the system is homogeneous in the prices; (2) the commodity equations are consistent and the relative prices are determined in the real sector; (3) a constraint operates in the monetary sector to determine the absolute level of prices.

[6] Cf., G. Cassel, *op. cit.*, p. 148, where he speaks of the pricing process as having "intrinsic consistency."

[7] G. Cassel, *ibid.*, p. 155. "In the general pricing problem, a multiplicative factor of all prices remains undetermined. The determination of this factor and, consequently, the final solution of the pricing problem, belongs to the theory of money."

[8] G. Cassel, *ibid.*, pp. 454–459.

II. AN ILLUSTRATION OF A CONSISTENT CLASSICAL SYSTEM

Despite its apparent elegance, the classical theory would be devoid of meaning if it were not possible to construct a system meeting the classical postulates. This is in fact Patinkin's position: "· · · it has not been possible to construct a system satisfying the classical dichotomy of determining relative prices in the real part of the model, and absolute prices through the money equation. It must be emphasized that *there is no monetary equation that we can use to remove this indeterminacy of the absolute prices.*"[9]

But despite Patinkin's denial it is easy to set up simple types of systems meeting the classical requirements. Consider, for example, a Casselian system in which we postulate only two commodities and money (the system may be generalized to $n-1$ commodities):

$$D_1 \equiv d_1 \frac{p_2}{p_1}; \qquad S_1 \equiv s_1 \frac{p_1}{p_2}; \qquad X_1 \equiv d_1 \frac{p_2}{p_1} - s_1 \frac{p_1}{p_2} = 0;$$

$$D_2 \equiv d_2 \frac{p_1}{p_2}; \qquad S_2 \equiv s_2 \frac{p_2}{p_1}; \qquad X_2 \equiv d_2 \frac{p_1}{p_2} - s_2 \frac{p_2}{p_1} = 0;$$

$$D_3 \equiv p_1 S_1 + p_2 S_2; \quad S_3 \equiv p_1 D_1 + p_2 D_2; \quad X_3 \equiv -p_1 X_1 - p_2 X_2 = 0.$$

The parameters of the system (the d_i and s_i) are normally taken to be positive.[10]

1. *The homogeneity assumptions.* It will be observed that the correct homogeneity assumptions hold, i.e., the commodity functions are homogeneous of degree 0 and the monetary functions are homogeneous of degree 1. By definition, the monetary equation is dependent upon the commodity equations, and is, therefore, redundant as an equilibrium constraint.

2. *The consistency assumption.* The system as it now stands does not contain enough information to permit us to say whether it is consistent. A necessary and sufficient condition that the system meet the classical assumption of consistency, however, is that the determinant of the matrix of the coefficients of X_1 and X_2 vanishes, i.e., that $s_2 = d_1(d_2/s_1)$. Making this substitution and solving X_1 (or X_2) for $p_1/p_2 = z = +\sqrt{d_1/s_1}$ insures that X_1, X_2, and X_3 vanish. Moreover, the values of D_1, D_2, S_1, and S_2 corresponding to z are the quantities of commodities demanded and supplied per unit of time in equilibrium. The *real* part of this system is therefore consistent and is completely determined by the price ratio z.

[9] D. Patinkin, *op. cit.*, p. 21. The italics appear in Patinkin's paper.

[10] The parameters are positive constants in a stationary economy and are positive, single-valued functions of time in a dynamic economy. For our purposes, either interpretation is permissible.

3. *The monetary constraint.* The *monetary* variables (D_3, S_3, p_1, and p_2) are still to be determined; but the entire set is determined once any variable of the set is known. Since every pair of prices (p_1^*, p_2^*) reducible to z is also a solution of the system, $(z, 1)$ is a solution. In particular, since $D_3(p_1, p_2) \equiv (1/\lambda)D_3(z, 1)$ it follows that D_3 (and hence S_3, p_1, and p_2) are determined up to a scale factor by substituting $(z, 1)$ for (p_1, p_2) in D_3. The Cambridge equation (e.1) shows that

$$M = KD_3(p_1, p_2) = K(1/\lambda)D_3(z, 1),$$

where M and K are parameters independent of p_1 and p_2.[11] Hence, $1/\lambda$ (or p_2) can be determined when z is given. When these operations are actually carried out, we find that $p_1 = p_2 z$, and $p_2 = (M/K)(d_1 + d_2 z)$, where $z = +\sqrt{d_1/s_1}$. Since d_1, d_2, s_1, and s_2 are all positive in the normal case, the solution is real and positive. The price level is therefore completely determined in the monetary sector by means of a single constraint.

III. PATINKIN'S ANALYSIS REVIEWED

Our model of a classical system contradicts Patinkin's position that it is impossible to construct systems of this type in which relative prices are determined in the real sector and the price level in the monetary sector. We shall now examine the details of his argument for the flaws which led him to believe that such systems could not be set up.

Patinkin's criticism of classical systems rests on two central propositions, which we shall examine *seriatim*. These propositions are: (1) that the assumption of a monetary constraint is logically incompatible with the assumption of homogeneity; and (2) that the homogeneity assumption contradicts the assumption of consistency.

1. Patinkin argues that the Cambridge equation (e.1) is incompatible with the homogeneity assumption, since in this equation D_n is homogeneous of degree 1 and M and K are homogeneous of degree 0.[12] That is to say, we cannot have

$$M \equiv KD_n(p) \tag{f.1}$$

for all prices, since this states that a function homogeneous of degree 0 is identically equal to a function homogeneous of degree 1. But this fact in no way invalidates the Cambridge equation, since it is a *constraint* (an equation of condition) and not an identity in the p's.

To put the matter another way, the fact that the Cambridge equation is nonhomogeneous is in no way incompatible with the fact that

[11] M and K may be considered either as constants (the stationary case) or as functions of time (the dynamic case).

[12] D. Patinkin, *ibid.*, p. 16.

its components are homogeneous functions of unequal degree. Indeed, inequality of degree of homogeneity of components is a property of all nonhomogeneous equations. When we say that $D_n(p)$ is a homogeneous function of degree 1, we mean simply that it is a function such that

$$(g.1) \qquad D_n(p) \equiv (1/\lambda)D_n(\lambda p)$$

for any set of prices and for any value of λ whatsoever. The *function* (g.1) is therefore defined for the class of all sets of $n-1$ prices reducible to the $n-2$ price ratios constituting the solution of the commodity equations. Only one set of this class, however, satisfies the Cambridge *equation* (e.1), and this set determines λ and the price level.[13]

2. Patinkin's second proposition that the homogeneity assumption contradicts the consistency assumption is not quite so easily disposed of, the reason being that he expresses himself ambiguously on this point. His position is stated in the following THEOREM: "*If the f_i and g_i ($i = 1, \cdots, n-1$) in* (6.1) *and* (6.2) *are independent and homogeneous of degree t' in all of the variables, then the system* (6.1)–(6.3) *is overdetermined. In particular, the Casselian system* (6.1)–(6.3) *is inconsistent.*"[14] We shall now show that under one interpretation this theorem is false, and under another it is trivial.

If the theorem is interpreted literally, it is false. The *functions* f_i and g_i in (6.1) and (6.2) cannot be independent if they are homogeneous of degree 0. Patinkin has shown (although for a different purpose) that the Jacobian of the system must vanish because of the homogeneity condition. But the vanishing of the Jacobian is a necessary and sufficient condition for functional dependence.[15] Hence, the functions f_i and g_i cannot be homogeneous of degree 0 and independent at the same time. The assumptions of the theorem are incompatible, not the postulates of classical systems.

We can, however, rephrase the theorem so that it is formally correct, as follows. THEOREM: *If the equations $X_i = 0$ ($i = 1, \cdots, n-1$) in* (a.5) *are independent and homogeneous of degree t' in all of the variables, then the system* (6.1)–(6.3) *is overdetermined. In particular, the Casselian system* (6.1)–(6.3) *is inconsistent.*

As we have seen, the $n-1$ equations (a.5) can be expressed in terms of $n-2$ price ratios by virtue of their homogeneity. Consider now the set of all systems of $n-1$ equations in $n-2$ unknowns. These systems

[13] Only *one* set of prices satisfies (e.1). The prices are linear functions of λ for a fixed set of price ratios. The right-hand member of (e.1) is linear in $1/\lambda$ for the same fixed set of price ratios. Hence (e.1) determines λ uniquely, and λ determines a unique set of prices.

[14] D. Patinkin, *ibid.*, p. 14.

[15] Ch.-J. de la Vallée Poussin, *Cours D'Analyse Infinitésimale,* Seventh Edition, New York: Dover Publications, Vol. II, Section 246, 1946.

can be divided formally into those that are (1) dependent and consistent, (2) independent and inconsistent, (3) dependent and inconsistent, and (4) independent and consistent. Evidently set (3) is empty, since by definition (see footnote 5) all dependent equations are consistent. Set (4) is explicitly ruled out by Patinkin at the beginning of his argument since he limits his analysis to those systems of equations that are inconsistent (overdetermined) when there are more independent equations than unknowns.[16] That is, in Patinkin's world, set (4) is also empty. This leaves us with only the first two sets in which dependence implies consistency and independence implies inconsistency. The theorem therefore simply states that inconsistent (i.e., independent) equations correspond to inconsistent systems, which is true but trivial. It in no way invalidates the classical assumption of consistency.

Patinkin's assumption of independence for classical systems is a curious one, the more so since he freely admits that in classical systems at least one of the original equations (the money equation) is dependent.[17] We have shown that the dependence may carry beyond this to the commodity equations. Although it is not necessary for the consistency of the commodity equations that they be dependent (see footnote 16), such dependence (and consistency) was guaranteed by the insertion of certain side conditions in several variants of the classical system. We turn now to an examination of the conditions under which such dependence may arise.

IV. THE DEPENDENCE OF THE COMMODITY EQUATIONS

Dependent equations are always consistent, but the converse does not follow. In the example presented earlier, the two commodity equations were made consistent by making the excess-demand functions *linearly dependent.* The same method may be used if a model of this type is extended to cover $n-1$ commodities.[18]

This type of dependence is rather special, although dependence of a similar type arises quite naturally out of certain economic settings. To illustrate how this may happen, we recall that a classical system is consistent if the solution of any $n-2$ commodity equations $X_i (i = 2, \cdots, n-1)$ involving $n-2$ price ratios is also a solution of X_1 or X_n (see c.1). Consider now a type of dependence in which X_1 can be expressed as a

[16] D. Patinkin, *op. cit.*, p. 4. For the sake of argument we tentatively accept Patinkin's restriction although we know of no economic or mathematical justification for it.

[17] J. R. Hicks, in *Value and Capital* (Oxford: Clarendon Press, 1946), makes the same assumption at one point (p. 60) but properly avoids it in the mathematical appendix to the same work (p. 314).

[18] The method is to make the determinant of the matrix of the coefficients vanish.

linear combination of powers of X_i ($i = 2, \cdots, n-1$), so that the following identity holds:

$$\text{(h.1)} \qquad \sum_{i=1}^{n-1} \phi_i(p) \cdot X_i^{(K)}(p) \equiv 0, \qquad (\phi_1 \neq 0, K \neq 0),$$

where the ϕ_i are functions only of the p's. If $K = 1$ and all of the ϕ_i are constants, the commodity equations are linearly dependent as exhibited in the example above.

A more important type of linear combination occurs when $K = 1$ and each $\phi_i = p_i$. This condition occurs in any barter economy, in any primitive monetary economy having only a money of account, and in a full monetary economy in which goods are sold only for the purchase of other goods. In these cases Say's Law holds, that is,

$$\text{(i.1)} \qquad \sum_{i=1}^{n-1} p_i X_i \equiv 0. \qquad (13.2)$$

By combining Say's Law with Walras' Law (c.1) we have

$$\text{(i.2)} \qquad X_n \equiv -\sum_{i=1}^{n-1} p_i X_i \equiv 0,$$

which shows that the vanishing of any $n-2$ equations necessarily implies the vanishing of the other two, if Say's Law holds.

Or again, if the quantity of money (M) is fixed exogenously by the state, and if the demand for cash balances (M_d) adjusts itself immediately to this fixed quantity through changes in the price level,

$$\text{(j.1)} \qquad M_d \equiv M.$$

Patinkin has shown that

$$\text{(j.2)} \qquad M_d - M \equiv D_n - S_n \equiv X_n, \qquad (8.6) \text{ and } (10.1)$$

from which it follows that

$$\text{(j.3)} \qquad X_n \equiv 0.$$

But if X_n vanishes, so does X_1, and the requirements for a consistent system are met. Moreover, it is easy to see that in this case also Say's Law holds; and, therefore, one of the commodity equations is a linear combination of the rest.

Toward the close of his analysis, Patinkin observed that Say's Law has the effect of reducing the number of independent equations to $n-2$, although he did not attempt to reconcile this fact with his earlier assumption of the independence (inconsistency) of the original set of $n-1$ equations. Instead he fell back upon the proposition that the price level would still be indeterminate since in his view there is no

monetary constraint to remove the indeterminacy. We have shown, however, that this proposition is false.

A consideration of the structure of (h.1) and of the examples falling under it should indicate that dependence may occur in a variety of ways.[19] There is, however, a more general property of classical systems that makes it unwise to assume a priori that the commodity equations are independent and inconsistent. As noted earlier, the functions from which these equations are formed are necessarily dependent; and hence a relation of the type

$$\psi(X_1, \cdots, X_{n-1}) \equiv 0$$

must hold between them. Assume now that we have a solution for $n-2$ of the commodity equations in $n-2$ price-ratios. Then

$$\psi(X_1^*, 0, \cdots, 0) \equiv 0,$$

where X_1^* is a constant (i.e., the value of X_1 for the solution). For a wide class of functions, the equation of dependence will imply that X_1^* is zero; that is, that the commodity equations are dependent and consistent. The fact that classical functions are dependent narrows down the field, so to speak, of systems of equations that can be independent.

On the other hand, the dependence of classical functions by no means eliminates the possibility that the equations may be inconsistent; and herein lies a most serious methodological dilemma. The classical school assumed that the system was consistent; and on logical grounds this is a perfectly admissible assumption. *It is equally, admissible, however, that the equations may be inconsistent.*

There appears to be no easy way out of this dilemma, or at least none that has been presented to date. One thing, however, is clear: the time-honored procedure of counting equations and unknowns is completely beside the point. In simple linear systems, an indeterminacy (an infinite number of solutions) can be detected in this way when the system contains more independent variables than equations. But in the theory of general market equilibrium too little is known as yet about the structure of the functions or about the domain of their parameters for this procedure to bear fruit. In the general case, we are simply given m equations in n unknowns; and we add nothing substantive to our knowledge if we count and find that $m = n$. The equations may have

[19] It should not be inferred from (h.1), however, that we are necessarily limited to dependence of this type. Consider, for example, the two commodity functions $X_1 = (p_1/p_2)^2 + (p_2/p_1)^2 - 2$ and $X_2 = (p_1/p_2) - (p_2/p_1)$, both being homogeneous of degree 0. The equation of dependence is $X_1 - X_2^2 \equiv 0$, which is not of type (h.1). Yet the vanishing of one of these functions implies the vanishing of the other.

one solution; they may have no solution; they may have an infinite number of solutions. And the solutions, if one or more of them exist, may be aceptable (in the sense that they contain no negative or complex quantities) or they may be unacceptable.

Moreover, the problem is not solved, although its dimensions may be reduced, if after counting we find that out of n equations in $n-2$ unknowns, no more than $n-2$ are independent. The remaining equations may be dependent or independent and in either case solutions may or may not exist.

We cannot, therefore, prove consistency by counting equations and unknowns. The converse, however, is equally true. We cannot, as Patinkin has attempted to do, prove inconsistency by counting equations and unknowns. The attempt to do this was in fact the basic methodological flaw underlying his argument.

National Bureau of Economic Research

THE CONSISTENCY OF THE CLASSICAL THEORY OF MONEY AND PRICES[1]

By Wassily Leontief

In his recent article on "The Indeterminacy of Absolute Prices in Classical Economic Theory,"[2] Don Patinkin arrives at the conclusion that "due to the traditional assumption that the demand for goods depends only on relative prices," the classical system as presented by Cassel is inconsistent. I have the impression that Mr. Patinkin's argument and consequently also his conclusion are erroneous. Since his criticism is aimed at the logical foundations of the non-Keynesian theory of general equilibrium, a further airing of the controversial issues involved can serve a useful purpose.

Patinkin considers a system involving n commodities of which the first $n-1$ represent real goods, the nth being paper money. The price of money p_n equals 1. The system consists of n demand equations:[3]

$$D_i = f_i(p_1, p_2, \cdots, p_{n-1}), \qquad (i = 1, \cdots, n); \tag{6.1}$$

n supply equations:

$$S_i = g_i(p_1, p_2, \cdots, p_{n-1}), \qquad (i = 1, \cdots, n); \tag{6.2}$$

and n equilibrium equations stating that the demand for each commodity equals its supply:

$$D_i = S_i, \qquad (i = 1, \cdots, n). \tag{6.3}$$

All in all there are $3n$ equations and $3n-1$ unknowns: n quantities supplied, n quantities demanded, and $n-1$ prices of real goods. Not all the equations, however, are independent. The demand for money D_n (expressed as a rate of flow per unit of time) is according to the classical theory identical with the total supply of real commodities, each multiplied by its respective price. Similarly, the supply of money S_n equals the aggregate values, i.e., quantities times prices, or all real demands. Thus in his equation (6.6) Patinkin shows that if the first $n-1$ equations in (6.3) are satisfied, the last, $D_n = S_n$, also necessarily holds true, i.e., it is not independent of the rest of the system and should not be regarded as a separate equilibrium condition. Its elimination reduces the total number of equations to $3n-1$, which can now be solved for the $3n-1$ prices of real commodities, the price of paper money, p_n, having been fixed in advance.

[1] This paper is the second of three papers relating to this general topic, all appearing in this issue.

[2] Econometrica, Volume 17, January, 1949, pp. 1–27.

[3] The numbers identifying various equations correspond to those used in Patinkin's article.

At a later stage of his argument, Patinkin substitutes the supply and demand equations (6.1) and (6.2) in the equilibrium equations (6.3) and thus reduces the original system to n excess-demand functions with $n-1$ unknown prices. If $D_i - S_i \equiv X_i$,

$$X_i(p_1, \cdots, p_{n-1}) = 0, \qquad (i = 1, \cdots, n). \tag{10.8}$$

The last of these can, according to the foregoing argument, be dropped as redundant.

Having thus presented what he considers to be the core of the classical general equilibrium theory, Patinkin proceeds to demonstrate its inconsistency. First of all he establishes the following mathematical THEOREM: *"If every equation of a system of K independent equations in K variables is homogeneous of some degree t in the same set of variables, then the system possesses no solution (i.e., it is inconsistent), with the possible exception of the one which sets each of the variables equal to zero"* (page 10).

Then he observes that according to the classical assumptions the $n-1$ supply and the corresponding $n-1$ demand functions for *real* commodities are homogeneous of degree zero in the $n-1$ prices, p_1, $p_2, \cdots, p_{n-1}$. That means that a simultaneous proportional increase of all these prices would leave all the quantities supplied and demanded unchanged. He furthermore shows that the equations describing the demand and supply for money are also homogeneous in the same prices but of the first degree; i.e., a proportional increase in prices would result in a same relative increase in demand and supply of money.

Having thus established that all the excess demand functions in system (10.8) are (according to the classical assumptions) necessarily homogeneous, Patinkin observes that "no matter what equation of (10.8) we drop (by virtue of their interdependence) we are left with $n-1$ independent equations in $n-1$ variables, where each of the equations is homogeneous in all the variables" (page 14). This being exactly a case referred to in the previously cited mathematical theorem, Patinkin concludes that "the Casselian system (6.1)–(6.3) is inconsistent."

Now I will show that Patinkin's argument, notwithstanding the system (6.1)–(6.3), is inconclusive. Having correctly indicated that the last, the monetary, equilibrium equation in (6.3) can be dropped since it is not independent of the other $n-1$ equations of the same set, he failed to notice that the classical system as presented above must necessarily contain a second redundant relationship: any one of the $2n-2$ individual supply and demand equations for *real* commodities [included in sets (6.1) and (6.2)] can be derived from the other $2n-3$ real supply and demand equations.

Since money does not enter in his utility, i.e., preference, function, an individual according to the classical theory of economic behavior offers real commodities and services for sale only in order to be able to purchase other real goods and services. This means that the shapes of the $2n-2$ functions describing his demand and supply for each of the real goods (as derived from his preference function) are necessarily interrelated in such a way that the unknown form of any one of them can be directly derived from the given shapes of the other $2n-3$.

Let us consider for example a simple system in which two competitively behaving individuals trade with each other two commodities, using paper money as a medium of exchange. If $D_1 = f_1(p_1/p_2)$ is the first individual's demand equation[4] for commodity 1, his supply equation for commodity 2 must necessarily be of the following shape: $S_2 = (p_1/p_2)f_1(p_1/p_2)$. This is so because *with any given* p_1 *and* p_2 his money receipts from the sale of commodity 2 must be always equal to his expenditures on commodity 1. Similarly, if $D_2 = f_2(p_1/p_2)$ is the other individual's demand for commodity 2, his supply of commodity 1 must be $S_1 = (p_2/p_1)f_2(p_1/p_2)$.

For the aggregate demand and supply of paper money we have

$$D_3 \equiv p_1S_1 + p_2S_2 = p_2f_2(p_1/p_2) + p_1f_1(p_1/p_2), \tag{6.4}$$

$$S_3 \equiv p_1D_1 + p_2D_2 = p_1f_1(p_1/p_2) + p_2f_2(p_1/p_2). \tag{6.5}$$

The complete set of excess-demand equations corresponding to Patinkin's system (10.8) appears now in the following form:

$$\begin{aligned} X_1 &\equiv D_1 - S_1 = f_1(p_1/p_2) - (p_2/p_1)f_2(p_1/p_2) = 0, \\ X_2 &\equiv D_2 - S_2 = f_2(p_1/p_2) - (p_1/p_2)f_1(p_1/p_2) = 0, \\ X_3 &\equiv D_3 - S_3 = [p_2f_2(p_1/p_2) + p_1f_1(p_1/p_2)] \\ &\qquad - [p_1f_1(p_1/p_2) + p_2f_2(p_1/p_2)] = 0. \end{aligned} \tag{10.8}$$

Patinkin correctly states that the last of these equations is a linear combination of the first two and thus does not represent an independent equilibrium condition. He is also obviously right in observing that the remaining two equations are homogeneous in p_1 and p_2. He erroneously assumes, however, that these two equations are independent of each other, and thus arrives at the conclusion that the system is overdetermined, i.e., contradictory. Actually the first two equations are linearly interdependent: $X_1p_1 = -X_2p_2$. Thus with positive prices, $X_1 = 0$ implies $X_2 = 0$, and vice versa. Far from being contradictory,

[4] I write the demand functions as $f_1(p_1/p_2)$ and $f_2(p_1/p_2)$ rather than in the more general form $f_1(p_1, p_2)$ and $f_2(p_1, p_2)$ in order to show that they are homogeneous of the degree zero in prices, p_1 and p_2.

the classical system can easily be solved for the price ratio p_1/p_2, and this solution as obtained from any one of the three excess-demand equations will necessarily satisfy also the other two.

It is not the purpose of this note to consider the numerous other criticisms of the classical and neoclassical theory contained in Patinkin's article. Insofar as some of them are directly or indirectly based on the argument discussed above, they must obviously stand and fall with it.

Harvard University

A NOTE ON PATINKIN'S "RELATIVE PRICES"[1]

By Cecil G. Phipps

It is maintained in the present note that the inconsistency of system B in Mr. Don Patinkin's recent article[2] is obtained by his use of a set of assumptions which can readily be shown to be contradictory. System B is the system in which stocks of money are assumed not to enter the utility functions.[3]

The first of these assumptions is made at the end of the summary appearing in the first article. There he states, "The assumption of perfect competition will be maintained throughout." In this statement, the freedom to trade without restraint is certainly implied.

On the same page he makes the second of these assumptions: "The individual maximizes (his) utility subject to (his) budget identity." While he does not say so specifically, it appears that this assumption carries over into all the other sections also.

From these two assumptions, perfect competition and maximization of utility, several useful corollaries may be derived. The first of these is this: *If a good has no utility for any trader, its price relative to a good which has marginal utility is zero.* The proof is based upon the implied behavior of the individual. Certainly, one who maximizes his utility would not exchange a useful good for any amount, however great, of a good which is useless both to himself and to every one else. Since no useful good would be exchanged for the useless good, the relative price of the useless good is zero. This good would not appear in the utility function of any individual.

The second corollary is very similar: *If a good has utility for some trader, but has no marginal utility for any trader, its price relative to the good which still has marginal utility is also zero.* This statement implies that the individual, while he has use for some of this good, now has all he wants of it. Indeed, more would be of no use and might even prove a detriment. Therefore, he would not trade a good for which he still has some use for more of one for which he has no further use. In this case also, the relative price of the good which has lost its marginal utility is zero. This situation is one in which our desire for a good is completely satisfied before the supply of the good is exhausted; e.g., the air we

[1] This is the third of three papers on this general topic, all of which appear in this issue.

[2] D. Patinkin, "Relative Prices, Say's Law, and the Demand for Money," Econometrica, Vol. 16, April, 1948, pp. 135–154. See also "The Indeterminacy of Absolute Prices in Classical Economic Theory," Econometrica, Vol. 17 January, 1949, pp. 1–27, by the same author.

[3] D. Patinkin, "Relative Prices . . . ," p. 145.

breathe out-of-doors. The good would appear in the utility function of those desiring it but the marginal utility expressed as a partial derivative would be zero.

The third corollary is the converse of the two above: *If the relative price of a good is zero, it has no marginal utility.* In the two cases above this statement has been true. Furthermore, these are the only two cases in which the partial derivative representing the marginal utility of a good will be zero. Among themselves, goods which have no utility have indeterminate price ratios. With these goods, one can only *play* at trading. Like statements are true of the excess supplies of the goods which have no marginal utility.

In section 2 of his first article, Patinkin introduces money into the problem. This he takes to be paper or token money concerning which he makes the assumption (page 140): "...people derive no ('direct') utility from paper money and therefore [the stock of money] does not enter the utility function." On the following page, he then makes the contradictory assumption that the price of money relative to useful goods is one. Using these contradictory assumptions, Patinkin finds it quite easy to show the system under discussion to be inconsistent. Whether or not it is correct to identify system B with the classical theory is open to question.

University of Florida

AN APPLICATION OF LEAST SQUARES TO FAMILY DIET SURVEYS[1]

BY M. H. QUENOUILLE

The application of the method of least squares is discussed for the case in which a large number of unknowns have to be estimated; in particular, the application to family diet surveys is considered. Methods are indicated for shortening the analysis and investigating the importance of different variables by a step-by-step procedure. Suggestions are made on the planning of surveys in the light of the methods investigated here.

SUMMARY

IN ANY well-planned diet survey, certain basic information is collected from each family observed. Briefly this includes (a) the total food consumption of the family during a given period, derived by weighing foods in stock at the beginning and end of the period and foods purchased during the period, (b) the age and sex distribution of the family and the occupations of all wage earners, and (c) the number of meals taken out by members of the family and the number of meals consumed within the household by visitors. (It is rarely possible to determine the exact outside consumption and consumption by visitors.)

From these data calculations are usually made of the nutritive value of the home diet in terms of energy value and of certain constituents, and of the requirements of the family for energy and the same constituents in terms of some accepted scale. Comparisons can then be made between population groups, usually with certain additional devices such as stratification by food expenditure per head of each family. Under the conditions prevailing up to 1939, the income per head of a family was the main determinant of food expenditure, and that in turn of quality of diet. Such a stratification therefore gave information valuable from the economic, sociological, and nutritional aspects. However, it is incapable of answering many subsidiary questions, such as the effects on food consumption of occupation, and family size and composition, except in so far as these tend to coincide with the economic classification.

When I was asked to undertake a study of the data from the Carnegie United Kingdom Dietary Survey (1937–39), the suggestion was made that I should explore to what extent additional information could be obtained about the effect of occupation and family composition on consumption, and whether or not any significant information could be derived about the consumption of individuals. Previously, only one

[1] My thanks are due to Dr. I. Leitch of the Commonwealth Bureau of Animal Nutrition for assistance in the preparation of this paper.

attempt had been made to assess individual consumption from family surveys, that of Clements [**1**], in which he gauged the individual consumptions by picking out and comparing families of similar constitution differing in one number. This was obviously unsatisfactory since it utilised only a small fraction of the available information and was subject to very large errors.

The Carnegie Survey was not planned with this type of refined analysis in view, but it has been found possible to obtain a great deal of significant information from the data. It would have been easier to analyse, and even more could have been extracted from it, had the survey been suitably planned.

Since the dietary survey records food purchases for the family as a unit, the results as referred to individuals will not necessarily give exactly the same answer as would individual dietary surveys. What is studied is the effect of family constitution and all the other variables on food purchasing habits. That is to say, the result represents a general pattern and not the findings for actual individuals.

At the same time, the form of analysis employed on the data of this survey indicates a method of dealing with least squares equations when a large number of variables are involved. Thus, in the following sections, a method is suggested of carrying out an analysis so that the major causes of variation can be removed and examined in a step-by-step procedure, and, in particular, the major causes of variation in family diet surveys are examined and examples are given.

1. INITIAL ANALYSIS

In general, we shall have a series of figures for the total consumption per family over a period of time, which figures it will be necessary to split into individual components. We can assign constants to each member of the family representing his or her daily consumption. Thus a_1 might be used to denote the consumption of a child, age 0–1, not breast-fed, and similarly constants may be used for other members of the family, say, as follows:

a_2	breast-fed child,	age 0–1,
b	child,	age 1–3,
c	child,	age 4–6,
d	child,	age 7–9,
e	child,	age 10–12,
f_1	male adolescent,	age 13–15,
f_2	female adolescent,	age 13–15,
g_1	male adolescent,	age 16–20,
g_2	female adolescent,	age 16–20,

m combined consumption of first male and female adults,
n consumption of each extra male adult,
p consumption of each extra female adult.

The age grouping is in accordance with the League of Nations scale of requirements, but any alternative grouping may be used. It must be remembered, however, that the use of additional constants requires a large amount of extra work and that this work would be in vain if the amount of data did not justify an extensive calculation. For this reason the grouping of the adults according to age will seldom be worthwhile, although it is not difficult to test this after a preliminary analysis. It should be noted that individual constants to represent adult male and female consumption separately have not been used. This is because the majority of families contain just one male adult and female adult member, and a deviation from this condition frequently represents a state of economic stress in the family. To base individual male and female figures on such cases would lead to spurious conclusions, although in this case the extra adults might be grouped according to age or some other classification to differentiate between two families living together and one family with more than two adult members. It is more important to classify the adult wage earners of the family according to the type of work that is being carried out. Thus instead of m, four additional constants, m_1, m_2, m_3, and m_4, might be used for the respective cases in which the male member is a heavy worker, medium worker, light worker, or unemployed. A similar classification might be used for n, but the relatively small number of cases in which families have more than one male adult member does not generally warrant the extra work involved.[2]

Using this system, for example, $m_2 + a_1 + 2c + d$ would represent one week's consumption by a family of one male medium worker, one adult female, and children aged 0–1, 4–6, 4–6, and 7–9. In the method of analysis proposed below, it will be presupposed that all families are observed for the same length of time. This is usually true, and a week is usually taken as the survey period, which implies payment of wages weekly and the usual absence of shopping facilities on Sunday.

In any analysis it will be necessary to take into account causes of variation other than family constitution. Five main causes other than family constitution exist:

Location. The manner in which the location of the family is taken into account will depend upon the importance that is placed upon district-to-district comparisons. Obviously if fine points are to be investigated, then a large number of observations must be taken and each district analysed

[2] Under present conditions, a large number of families are sharing accommodations and food, and this would no longer be true.

separately, but in most cases the object will be to find the difference in consumption per individual. For this purpose it will suffice to add constants representing the relative individual consumption for each district or type of district. For example, if we group the districts into rural, light industrial, and heavy industrial, constants u, v, and w, $u + v + w = 0$, might be used to represent the relative individual consumptions and thus $m_2 + a_1 + 2c + d + 6v$ would represent the above family in a light industrial district. It will be seen that this representation fails in that it assigns only one constant to each district comparison irrespective of age or occupation. To be most effective these constants should take into account differences in age and occupation; but, provided that the families used in each analysis are of roughly the same size, the error introduced will be insignificant if we remember that adults will account for more, and children less, than is indicated by the values that are deduced.

Food expenditure or income group. Allowance can be made for economic differences in the same manner as for differences in location, but the extreme differences in this case are so large that it would seem advisable, if the data permit, to analyse them in more than one portion, although this is a difficulty that can be overcome by the use of further constants.

Size of family. The variation of individual intake with size of family might be expected, and this might be studied by the introduction of further constants (see below), but the increasing variation in family consumption with size of family necessitates separate analyses for increasing size of family, prior to any over-all analysis, to determine the relative accuracy of results obtained from families of different sizes. It would seem likely that in most cases the variation would increase with size of family, leaving the coefficient of variation constant; and in some recent unpublished work on calorie intake this was in fact true, the coefficient of variation remaining at roughly 15% throughout when all definable causes had been eliminated. Thus the results from families of different sizes must be weighted according to some preliminary analysis or some predetermined scale before a final analysis is undertaken.

Seasonal variation. The seasonal variation in consumption is relevant to any analysis, particularly of fruit and vegetables, but does not cause appreciable trouble in the analysis, although this effect will often vary with size of family. Usually in the planning of the survey this effect will be made orthogonal to most other effects.

Visitors and meals out. The manner in which visitors and meals out are dealt with must depend upon what is intended to be measured by the survey. If it is the individual consumption at home that is to be measured then no adjustment for meals out is necessary, and such an adjustment would seem generally undesirable, especially when rationing

is in force. An adjustment for visitors can be made, but unfortunately the case of the casual visitor (when further food is introduced into the house) cannot be classified separately from the case of the "unexpected caller" (when little or no extra food is introduced) or of the neighbor who returns the compliment. Some distinction is preferable, although, since any constant will take on an average value for the effects of different types of visitors and since the proportion of visitors is in any case usually small, such a distinction is not important.

2. THE BASIC METHOD OF ANALYSIS

When the "model" for analysis has been specified, then it is next necessary to determine the form and method of analysis. The analysis can be carried out conveniently by the method of least squares, which minimizes $\sum \omega_i(x_i - X_i)^2$, where x_i is the mean daily consumption of the ith family, X_i is the expected daily consumption of the ith family in terms of unknown constants, and ω_i is the weight attached to the observation depending on the size of family and possibly the length of observation.

This well-known method gives rise to as many simultaneous linear equations as there are unknown constants, and, if the accuracy of the estimated constants is to be found, a matrix of the same order must be inverted. It is for this reason that it is necessary to reduce the number of constants to a minimum. The calculation, however, can be reduced since the unknowns can be regarded as several sets of constants. For example, either m_1, m_2, m_3, or m_4 occurs in every family and, similarly, either u, v, or w occurs in every income group classification. With two such sets it has been shown [2] that an analysis of variance can be carried out on one set and adjusted by covariance on a dummy variate representing the other. This can similarly be carried out with three or more sets, although it requires the use of covariances on covariances which will be discussed later. However this suggests methods of planning a survey so that the subsequent statistical analysis can be greatly shortened. The simplest method would be to ensure that district, income group, and type of worker comparisons are all orthogonal by taking the same proportions of each type of worker in each district and income group and by taking the same proportions of each income group from each district, or, if each income group is to be analysed separately, by taking the type of worker and district orthogonal to each other. Unfortunately this is not always practicable because the interdependence of the factors involved makes the collection of such samples difficult in many cases, unreal, and inefficient for the purpose of subsequent over-all comparisons. The alternative, which makes a compromise between the statistical analysis and the efficiency of individual compari-

sons on the one hand, and the ease of sampling and the efficiency of over-all comparisons on the other, seems to demand the use of "partially-orthogonal" samples. An approach to such samples can be made by the covariance method mentioned above. When there are two criteria of classification then an analysis of one criterion with covariance on dummy variates representing the other will involve a covariance with one less variable than is involved in the second criterion. However this covariance can be partitioned in different ways into orthogonal components, and if the majority of these components are taken to be orthogonal to the first criterion, then these can be taken out in the initial analysis and the covariance carried out on the remaining components. For example, suppose that the second criterion splits the data into three groups, u, v, w, say, and that the number of observations in the subgroups are

TABLE I

Second Criterion	Groups of First Criterion					
	1	2	3	4	5	...
u	n_1	n_2	n_3	n_4	n_5	...
v	m_1	m_2	m_3	m_4	m_5	...
w	$m_1p - n_1$	$m_2p - n_2$	$m_3p - n_3$	$m_4p - n_4$	$m_5p - n_5$	...

those given in Table I. It is seen that an initial analysis of the first criterion and of $u + w$ versus v can be carried out and adjusted by a single covariance using the dummy variate ξ taking values 1 for u, 0 for v, and -1 for w.

When three or more criteria of classification are used then the same principle can again be employed, although its application is more involved and depends upon whether or not the several criteria are to be taken as mutually orthogonal. For example, if a third criterion with the grouping x, y, z, is employed with numbers in the subgroups as in Table I, then $u + w$ and v must be orthogonal to $x + z$ and y for a second dummy variate to be employed.

A final simplification in the analysis that might be employed is the assumption that the consumption by the children behaves in a "regular" manner, i.e., its differentials exist everywhere and it can be adequately represented by a polynomial. This is not always possible, but it provides a useful simplification, especially when, as is common, the estimates in successive age groups tend to be negatively correlated. In any case, some care must be exercised in using this method to ensure that the representation is adequate, and if the sample is large there would seem little point in its use. In the following example this method has been used

since the purpose is to demonstrate the analysis, and on as small a sample as sixty observations the estimates of the constants would be subject to very large error.

Example. To demonstrate the analysis, sixty observations, C, of calorie consumption of families with five children and spending about the same amounts on food per head were taken from a pre-war survey. Originally

TABLE II

Type of Worker	Type of District								
	Heavy Industrial $\xi = -1$			Light Industrial $\xi = 0$			Rural $\xi = 1$		
	l	q	C	l	q	C	l	q	C
Heavy	−3	15	12685	−5	9	15712	3	19	13668
	−2	6	14513	0	6	15056	−7	15	16639
	−7	15	11478	3	19	15314	−3	11	14758
	−1	9	14867	−3	15	12844	−5	9	15451
	−13	45	53543	−5	49	58296	−12	54	60516
Medium	−6	14	11868	1	11	16466	2	10	15447
	−1	11	12194	−1	7	14291	−1	7	15970
	1	9	15659	−3	15	14527	−2	4	14158
	−3	15	14016	−3	15	12480	−6	10	16034
	1	9	13343	0	10	17569	−1	7	17737
	−3	7	11391	−7	15	12757	−7	15	15450
	−5	9	13410	2	14	18002	−6	10	15444
	5	15	17092	−4	10	15144	−1	11	16478
	2	14	15118	−3	7	13275	−1	6	19622
				−6	10	14100	−5	9	20552
							−3	5	19830
	−9	103	124091	−24	114	148611	−31	94	186717
Light	−6	14	11777	−1	7	15416			
	−3	7	14200	−1	11	20989			
	0	8	18300						
	1	11	13893						
	−8	40	58170	−2	18	36405			
Unemployed	−8	16	12102	−6	14	15246	−4	18	11809
	−1	15	11340	−3	13	13166	−4	10	16166
	−2	6	13425	−2	6	17502			
	−4	16	14057	5	15	16275			
	−7	15	13658						
	5	15	17667						
	−17	83	82296	−6	48	62189	−8	28	27975
Total.......	−47	271	318100	−37	229	306131	−51	176	275208

there had been about eighty observations falling into this category, but observations on families with breast-fed children, pregnant women, or more than two adults, were rejected, and a further group of 17 observations was rejected to bring the remainder into a pattern similar to that indicated in Table I. In this form the equations to be solved involve only one constant instead of the seven, m_1–m_4, u, v, and w. Since the number of children in each family is the same, no comparison of adult consumption with consumption per child is possible, and the small number of observations precludes all but the grossest comparisons. Thus

TABLE III

PRELIMINARY ANALYSES OF COVARIANCE

	Degrees of Freedom	Sum of Squares of C	Sum of Squares of ξ	Sum of Products of ξ and l	Sum of Products of ξ and q	Sum of Products of ξ and C
Mean	1	13,483,175,245	0.6000	13.5000	−67.6000	−89,944
Workers	3	15,278,930	3.5333	− 0.7676	− 3.3333	35
$w + u - 2v$	1	2,993,784	0.3000	1.2000	0.5500	948
Residual	55	288,369,561	35.5676	−17.9333	−24.6176	46,069
	60	13,789,817,520	40.0000	−4.0000	−95.0000	−42,892
				Sum of Squares of l	Sum of Products of l and q	Sum of Products of l and C
Mean	1			303.750	−1521.000	−2023738
Workers	3			4.533	− 19.883	8035
$w + u - 2v$	1			4.800	2.200	3791
Residual	55			609.917	2.683	125079
	60			923.000	−1536.000	−1886833
					Sum of Squares of q	Sum of Products of q and C
Mean	1				7616.267	10133679
Workers	3				100.517	− 38244
$w + u - 2v$	1				1.008	1737
Residual	55				760.208	−131110
	60				8478.000	9966062

If the predicted calorie consumption is given by $a_1\xi + a_2l + a_3q$, then, from Table III,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 35.667 & -17.933 & -24.617 \\ -17.933 & 609.917 & 2.683 \\ -24.617 & 2.683 & 760.208 \end{bmatrix}^{-1} \begin{bmatrix} 46069 \\ 125079 \\ -131110 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0291991 & 0.0008544 & 0.0009425 \\ 0.0008544 & 0.0016646 & 0.0000218 \\ 0.0009425 & 0.0000218 & 0.0013459 \end{bmatrix} \begin{bmatrix} 46069 \\ 125079 \\ -131110 \end{bmatrix} = \begin{bmatrix} 1328.475 \\ 244.709 \\ -130.311 \end{bmatrix}.$$

The analysis of covariance is then completed as follows:

	Degrees of Freedom	Sum of Squares of C	Mean Square	Standard Deviation
Regression	3	108,894,891		
Residual (2)	52	179,474,670	3,451,436	1,858
Residual (1)	55	288,369,561		

We may now test different effects, for example,

(1) Standard error of $a_1 = [(0.0291992)\ (3,451,436)]^{1/2} = 317.5$,
Standard error of $a_2 = [(0.0016646)\ (3,451,436)]^{1/2} = 75.8$,
Standard error of $a_3 = [(0.0013459)\ (3,451,436)]^{1/2} = 68.2$.

(2) Difference in calorie intake of children, aged 16–20 and 7–9 equals $3a_2 + 9a_3 = -438.7$.
Standard error of difference equals $[(9\cdot 0.0016646 + 54\cdot 0.0000218 + 81\cdot 0.0013459)\ 3,451,436]^{1/2} = 688.4$

(3) Difference between mean calorie intake of medium and unemployed workers, adjusted for differences in family constitution and district equals $15314 - 14372 - (24a_1 + 27a_2 - 173a_3)/60 = -75$.
Standard error of difference equals $[(1/30 + 1/12 + 48.06/3600)\ 3,451,436]^{1/2} = 669.8$.

the substitutions

$$a_2 = d - 3l + 9q, \qquad b = d - 2l + 4q, \qquad c = d - l + q,$$

$$e = d + l + q, \quad f_1 = f_2 = d + 2l + 4q, \quad q_1 = q_2 = d + 3l + 9q,$$

were made. Table II gives the mean daily consumption per family, together with the coefficients of l and q for each family.[3] The covariances

[3] My thanks are due to Dr. D. P. Cuthbertson of the Rowett Institute for Research in Animal Nutrition for permission to use the data in this table from the Carnegie Dietary Survey prior to the publication of the analysis of this survey.

are then analysed in Table III and in the material which follows immediately thereafter. The analysis thus follows the normal course and standard errors can be attached to the resulting estimates and their differences in the normal way. This analysis, although crude in many ways, demonstrates on a smaller scale how the method works, and it also emphasizes a difficulty that will often occur: Is the "model" used for the analysis sufficiently good, i.e., what additional factors, if any, should be introduced and how can they be introduced with a minimum of labour?

3. COVARIANCES ON COVARIANCES

The above analysis was made without undue difficulty because it was assumed that all the possible influencing factors had been taken into account and that the number of such factors was relatively small. However, in practice this is not always true and, as pointed out in the last section, we are frequently uncertain whether or not additional variables should be added when the number of variables is very large. Consequently, although the above analysis saves a great deal of time, the computation may still be very large. For example, we may have twenty prime causes of variation and another ten possible causes. By preliminary planning, ten of the prime causes may be made orthogonal, but a covariance analysis still involves the inversion of a 10 by 10 matrix with further inversions if the possible causes of variation are also to be tested. This work can again be shortened using the "covariance-on-covariance" technique. Thus, suppose that the residuals in the initial analysis of variance are represented by y and that the initial covariance on $x_1, \cdots, x_p$ estimates the relation $Y = a'x$, where $a' = [a_1, \cdots, a_p]$ and $x' = [x_1, \cdots, x_p]$. Let $s(\xi_i\eta)$ represent the column vector with elements $\sum \xi_i\eta$ and let $S(\xi_i\eta_j)$ represent the matrix with elements $\sum \xi_i\eta_j$. Also let $S^{-1}(\xi_i\eta_j) = C(\xi_i\eta_j)$; then $a = C(x_ix_j)s(x_iy)$.

If further variables $z_1, \cdots, z_q$ are now introduced, then regressions on $x_1, \cdots, x_p$ give $Z_k = S'(z_kx_i)\,C(x_ix_j) = l_k'x$, say, and the residual sum of squares and products for $y, z_1, \cdots, z_q$ when $x_1, \cdots, x_p$ have been eliminated are given by formulae such as

$$\begin{aligned}
\Sigma_x y^2 &= \Sigma y^2 - S'(x_iy)C(x_ix_j)S(x_iy) = \Sigma y^2 - a'S(x_iy),\\
\Sigma_x z_1^2 &= \Sigma z_1^2 - S'(x_iz_1)C(x_ix_j)S(x_iz_1) = \Sigma z_1^2 - b_1'S(x_iz_1),\\
\Sigma_x yz_1 &= \Sigma yz_1 - S'(x_iy)C(x_ix_j)S(x_iz_1) = \Sigma yz_1 - a'S(x_iz_1)\\
&= \Sigma yz_1 - b_1'S(x_iy),
\end{aligned}$$

where the subscript x is used to indicate that the variables $x_1, \cdots, x_p$ have been eliminated. Thus, if a further covariance on $z_1, \cdots, z_q$ is carried out, the inverse matrix is $C_x(z_iz_j)$ and we now have

$$Y_x = S_x'(yz_i)C_x(z_iz_j)z_x = d'z_x,$$

and the residual sum of squares for y is given by

$$\Sigma_{xz}y^2 = \Sigma_x y^2 - d'S_x(yz_i).$$

If we write out the relation connecting Y with $x_1, \cdots, x_p, z_1, \cdots, z_q$, we now get

$$Y = d'z + (a' - dB)x = d'z + e'x,$$

where

$$B' = [b_1, b_2, \cdots, b_q] = S'(z_k x_i)C(x_i x_j).$$

The covariance matrices of $d_1, \cdots, d_q$ and $a_1, \cdots, a_p$ are $C_x(z_i z_j)$ and $C(x_i x_j)$, respectively, so that the covariance matrix of $e_1, \cdots, e_p$ is

$$C(x_i x_j) + B'C_x(z_i z_j)B$$

and the covariance of $d_1, \cdots, d_q$ with $e_1, \cdots, e_p$ is given by

$$-B'C_x(z_i z_j).$$

Thus the over-all covariance matrix of $e_1, \cdots, e_p, d_1, \cdots, d_q$ is

$$\begin{bmatrix} C(x_i x_j) + B'C_x(z_i z_j)B & \cdots & -B'C_x(z_i z_j) \\ \cdots\cdots & \cdots & \cdots\cdots \\ -C_x(z_i z_j)B & \cdots & C_x(z_i z_j) \end{bmatrix}.$$

The importance of this approach, which effectively extends a formula given by Cochran [3] for the addition of an extra variable in a regression, is that the over-all analysis involving $p + q$ variables requires the inversion of only two matrices of orders p and q. The analysis is conveniently carried out in covariance form, and can easily be extended since the calculation of the over-all covariance matrix effectively brings us back to a single covariance and the process can be repeated again. In multiple regression terminology these formulae are equivalent to the statement that a regression of y on $x_1, \cdots, x_p, z_1, \cdots, z_q$, can be carried out by calculating initial regressions of $y, z_1, \cdots, z_q$ on $x_1, \cdots, x_p$ and subsequently a regression of the residuals of y on the residuals of $z_1, \cdots, z_q$ when $x_1, \cdots, x_p$ have been eliminated.

This analysis may be conveniently split up into the following calculations:

(1) the sums of squares and products of all variables,
(2) the inverse matrix $C(x_i x_j)$,
(3) the coefficients b_k and the matrix B,
(4) the matrices $S_x(z_i y)$ and $S_x(z_i z_j)$,
(5) the inverse matrix $C_x(z_i z_j)$,
(6) the coefficients d and e and the residual $\sum_{xz} y^2$, and
(7) the over-all covariance matrix.

Step (1) would be carried out whatever the method of analysis, while steps (3), (4), and (6) are roughly equivalent to the multiplication that would be required after the calculation of an over-all inverse matrix. Thus the above calculation effectively replaces the inversion of an over-all matrix by steps (2), (5), and (7), and the inversion of, say, a 12 by 12 matrix can be replaced by the inversion of two 6 by 6 matrices together with three matrix multiplications.

TABLE IV

TEST OF THE DIFFERENCE BETWEEN MALE AND FEMALE ADOLESCENT CONSUMPTION

	Degrees of Freedom	Sum of Products of ξ and s	Sum of Products of l and s	Sum of Products of q and s
Mean	1	−0.500	−11.250	56.333
Workers	3	0.833	− 1.417	4.500
$w + u - 2v$	1	0.050	0.200	0.092
Residual	55	−5.383	26.467	−12.925
	60	−5.000	14.000	48.000
		Sum of Products of C and s	Sum of Squares of s	
Mean	1	74953.2	0.41667	
Workers	3	−1252.5	2.75000	
$w + u - 2v$	1	158.0	0.00833	
Residual	55	−2232.7	21.82500	
	60	71626.0	25.00000	

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0.0291991 & 0.0008544 & 0.0009425 \\ 0.0008544 & 0.0016646 & 0.0000218 \\ 0.0009425 & 0.0000218 & 0.0013459 \end{bmatrix} \begin{bmatrix} -5.383 \\ 26.467 \\ -12.925 \end{bmatrix} = \begin{bmatrix} -0.146758 \\ 0.039175 \\ -0.021892 \end{bmatrix}$$

Residual sum of products of C and s (corrected for ξ, l, and q) equals $-2232.7 - 1009.3 = -3242.0$.
Residual sum of squares of s (corrected for ξ, l, and q) equals $21.82500 - 2.10983 = 19.71517$.
Regression coefficient d is $-3242.0/19.71517 = -0.0507224(3242.0) = -164.4$.

$$-0.0507224\ \mathrm{B} = \begin{bmatrix} 0.0074439 \\ -0.0019870 \\ 0.0011104 \end{bmatrix}, \quad 164.4\ \mathrm{B} = \begin{bmatrix} -24.1 \\ 6.4 \\ -3.6 \end{bmatrix}, \quad s = \begin{bmatrix} 1304.4 \\ 251.1 \\ -133.9 \end{bmatrix}$$

$$0.0507224\ \mathrm{B}'\mathrm{B} = \begin{bmatrix} 0.0010925 & -0.0002916 & 0.0001630 \\ -0.0002916 & 0.0000778 & -0.0000435 \\ 0.0001630 & -0.0000435 & 0.0000243 \end{bmatrix}$$

Thus the covariance matrix for e_1, e_2, e_3, and d is

$$\begin{bmatrix} 0.0302916 & 0.0005628 & 0.0011055 & 0.0074439 \\ 0.0005628 & 0.0017424 & -0.0000217 & -0.0019870 \\ 0.0011055 & -0.0000217 & 0.0013702 & 0.0011104 \\ 0.0074439 & -0.0019870 & 0.0011104 & 0.0507224 \end{bmatrix}.$$

In practice the inversion of anything greater than an 8 by 8 matrix is tedious. The 3 by 3 matrix would seem to be a convenient unit with which to work, although when we are uncertain whether certain factors should be taken into account these may be tested and, if necessary, incorporated.

A further point that should be noted is that the set of orthogonal factors eliminated by the analysis of variance can be considered as the first step in the above process and that, correspondingly, these factors can be incorporated into the over-all inverse matrix at any stage.

Example. To demonstrate the form of this analysis, we might test whether the accuracy of the analysis carried out in Tables II and III might be improved by differentiating between the sexes of children over 13

and whether a reliable estimate of any such differences can be obtained. This is done using a dummy variate ξ which takes values $+1$ for male adolescents, -1 for female adolescents, and 0 for other children. The form of the analysis is shown in Table IV. The estimated regression coefficient in this case is -164.4 ± 421.9 so there is no gain from introducing this extra variable, the accuracy of which is not very good. However, it has been assumed that this extra variable has been taken into account and an over-all covariance matrix has been calculated.

One further point should be noted about the form of analysis: the individual analyses of variance reveal wherever two sets of factors in the collected data deviate very greatly from orthogonal samples and therefore indicate where the existence of interaction will be of greatest importance.

4. DETERMINATION OF THE WEIGHTS

It has already been pointed out that an increase in size of family normally leads to an increase in the variation and that for the method of least squares to be efficient the observations should be weighted according to their relative accuracy. This can be done according to some predetermined scale or according to a rough initial analysis. Fortunately it has been shown [4] that the analysis is accurate even if the weighting is fairly rough. However some care is necessary in carrying out the over-all analysis because of the problems raised by interaction with size of family. Thus, for example, it is fairly obvious that the effect of location and economic classification will vary with size and constitution of family. Normally the assumption that these effects are constant for families of a particular size will be sufficiently accurate, but any further assumption concerning their variation with size of family will lead to a greater degree of inaccuracy in the representation. Again, if the individual consumption varies with size of family this tends to give an incorrect picture in an over-all analysis. Thus, for example, if the relative consumption decreases with size of family, since the larger, i.e., older, families contain a greater proportion of adolescents, this effect, if not taken into account, will be shown in an over-all analysis by a decrease in the estimated consumption of the adolescents.[4] Thus the following approach is suggested.

[4] There are, however, two other points which should be noted at this stage. Firstly, if interaction is detected in analyses by size of family, the combination of these analyses cannot be carried out uncritically if the results are to be of general application. Thus, if children in small families are getting more than their requirements and children in large families less than their requirements, to state that the average is "just right" hardly does justice to the situation, while a standard error attached to such an average acquires a specialized meaning. Secondly, an effect of the kind described above can occur irrespective of size of family if there is a tendency towards "arrested development" in the family consumption.

As a first step in the analysis we eliminate effects that are known to vary with size of family. A rough analysis is then carried out eliminating the effect of family constitution. This is most conveniently carried out using graduated parameters, as in the above example. Unless great accuracy is required, the type of work need not be taken into account since the determination of the weights to within 20% will usually suffice, although if it is desired to investigate the extent and form of interaction larger analyses will be required. The relative weights given by the reciprocals of the residual mean squares can now be used to combine the individual analyses *after* the stage where the effects varying with size of family have been eliminated. The analysis then follows the usual lines with the initial matrix $C(x_i x_j)$ now taking the form:

$$\begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & C_n \end{bmatrix}$$

However, it may subsequently be necessary to combine or analyse the estimates obtained by size of family, so that some general remarks on the combination of least-squares estimates are necessary.

5. THE COMBINATION OF LEAST-SQUARES ESTIMATES

If we have a series of normally and independently distributed estimates $\theta_1, \theta_2, \cdots, \theta_n$, of a parameter θ, with variances $V_1, V_2, \cdots, V_n$, then it is well-known that the most efficient combined estimate $\bar{\theta}$ is given by $(\sum V_i^{-1})\bar{\theta} = \sum V_i^{-1}\theta_i$, with a variance $\bar{V}$ given by $\bar{V}^{-1} = \sum V_i^{-1}$. Similarly, if we have a series of normally and independently distributed estimates $\theta_1, \theta_2, \cdots, \theta_n$, of a set of parameters θ, with covariance matrices $V_1, V_2, \cdots, V_n$, then the most efficient combined estimates are given by $(\sum V_i^{-1})\bar{\theta} = \sum V_i^{-1}\theta_i$, with a covariance matrix $\bar{V}$ given by $\bar{V}^{-1} = \sum V_i^{-1}$. This, in effect, states that an over-all least-squares analysis should be carried out with weighting proportional to the residual mean squares in the individual analyses. An alternative approach to this seems worth noting. If we have two sets of estimates θ_1, θ_2 with covariance matrices V_1, V_2, then we can find a transformation A such that $A'V_1A$ and $A'V_2A$ are diagonal and, correspondingly, so that θ is transformed to $A'\theta$. The transformed set of estimates are independent of each other so that the ordinary weighting rules will apply and

$$[(A'V_1A)^{-1} + (A'V_2A)^{-1}]A'\bar{\theta} = (A'V_1A)^{-1}A'\theta_1 + (A'V_2A)^{-1}A'\theta_2,$$

i.e.,

$$A^{-1}(V_1^{-1} + V_2^{-1})\bar{\theta} = A^{-1}(V_1^{-1}\theta_1 + V_2^{-1}\theta_2),$$

which is the same formula as obtained previously. This alternative ap-

proach, however, shows that the loss of information resulting from the direct combination of least-squares estimates will depend upon the deviations of V_1 and V_2 from diagonal matrices. In particular, the deviation of $V_1^{-1}V_2$ or $V_2^{-1}V_1$ will largely determine the loss of accuracy for the combination of matrices. Also it suggests that an initial transformation to make the matrices V_1 and V_2 approximately diagonal might be fairly efficient.

As a first step in investigating the efficiency of direct combination we might consider least-squares matrices of the kind $A_1P_1A_1$ where A_1 is diagonal and

$$P_1 = \begin{bmatrix} 1 & p_1 & p_1 \cdots p_1 \\ p_1 & 1 & p_1 \cdots p_1 \\ p_1 & p_1 & 1 \cdots p_1 \\ \cdots & \cdots & \cdots \\ p_1 & p_1 & p_1 \cdots 1 \end{bmatrix}.$$

This type of matrix will arise when each observable has an equal probability of occurrence with every other observable. The inverse of this matrix is $A_1^{-1}P_1^{-1}A_1^{-1}$, where

$$P_1^{-1} = \begin{bmatrix} P_1 & Q_1 & Q_1 \cdots Q_1 \\ Q_1 & P_1 & Q_1 \cdots Q_1 \\ Q_1 & Q_1 & P_1 \cdots Q_1 \\ \cdots & \cdots & \cdots \\ Q_1 & Q_1 & Q_1 \cdots P_1 \end{bmatrix},$$

$$P_1 = \frac{1 + p_1(n-2)}{(1-p_1)[1+p_1(n-1)]},$$

$$Q_1 = -\frac{p_1}{(1-p_1)[1+p_1(n-1)]},$$

and n is the order of the matrix. This matrix can now be used to investigate the loss of information. Thus, if $A_1 = a_1^{\frac{1}{2}}I$, so that the least-squares matrix is a_1P_1, and the estimates from this matrix are directly combined with those from a second matrix a_2P_2, then the percentage loss of information is found to be

$$\frac{100(n-1)(p_1-p_2)^2}{(1-p)\{1+p(n-1)\}\{1+p_1(n-2)\}\{1+p_2(n-2)\}} \cdot \frac{a_1a_2}{(a_1+a_2)^2},$$

where

$$p = \frac{a_1p_1 + a_2p_2}{a_1+a_2}.$$

The latter half of this expression depends only upon the relative

number of observations in the two groups and cannot take a value exceeding $\frac{1}{4}$ which it does when $a_1 = a_2$. Table V gives the values of this expression for $a_1 = a_2, 2a_2$; $n = 3, 5, 7, 9$; $p_1 = 0.0, 0.2, 1.0$; $p_2 = 0.0, 0.2, 1.0$. It is apparent that the loss of information is serious only when the difference between p_1 and p_2 is large, i.e., in effect, when one set of observations might be used to offset a high correlation between estimates obtained from the other. This effect is naturally more important if the

TABLE V

PERCENTAGE LOSS OF INFORMATION RESULTING FROM THE DIRECT COMBINATION OF LEAST-SQUARES ESTIMATES

p_2 \ p_1	$a_1 = a_2$						$a_1 = 2a_2$					
	0.0	0.2	0.4	0.6	0.8	1.0	0.0	0.2	0.4	0.6	0.8	1.0
n = 3												
0.0	0.0	1.5	5.1	10.0	16.5	25.0	0.0	1.3	4.5	9.3	16.4	28.6
0.2	1.5	0.0	1.1	3.9	8.3	15.1	1.4	0.0	1.0	3.6	11.4	18.0
0.4	5.1	1.1	0.0	0.9	3.6	8.9	4.6	0.9	0.0	0.8	4.7	11.0
0.6	10.0	3.9	0.9	0.0	0.9	4.8	8.9	3.4	0.8	0.0	0.9	6.2
0.8	16.5	8.3	3.6	0.9	0.0	1.9	14.1	6.9	2.9	0.8	0.0	2.6
1.0	25.0	15.1	8.9	4.8	1.9	0.0	20.0	11.5	6.5	3.4	1.4	0.0
n = 5												
0.0	0.0	2.0	5.0	8.3	12.1	16.7	0.0	1.7	4.3	7.3	11.4	18.2
0.2	2.0	0.0	0.7	2.3	4.4	7.3	1.9	0.0	0.6	2.1	4.3	8.5
0.4	5.0	0.7	0.0	0.4	1.6	3.6	4.9	0.7	0.0	0.4	1.6	4.3
0.6	8.3	2.3	0.4	0.0	0.4	1.7	7.9	2.0	0.4	0.0	0.4	2.1
0.8	12.1	4.4	1.6	0.4	0.0	0.6	11.0	3.8	1.3	0.3	0.0	0.8
1.0	16.7	7.3	3.6	1.7	0.6	0.0	14.4	5.8	2.7	1.2	0.4	0.0
n = 7												
0.0	0.0	2.1	4.5	6.9	9.4	12.5	0.0	1.7	3.7	5.9	8.7	13.3
0.2	2.1	0.0	0.5	1.5	2.7	4.3	2.0	0.0	0.4	1.3	2.6	4.9
0.4	4.5	0.5	0.0	0.2	0.9	1.9	4.5	0.5	0.0	0.2	0.9	2.3
0.6	6.9	1.5	0.2	0.0	0.2	0.9	6.8	1.3	0.2	0.0	0.2	1.1
0.8	9.4	2.7	0.9	0.2	0.0	0.2	8.9	2.5	0.7	0.2	0.0	0.4
1.0	12.5	4.3	1.9	0.9	0.2	0.0	11.1	3.5	1.4	0.6	0.2	0.0
n = 9												
0.0	0.0	2.1	4.0	5.8	7.7	10.0	0.0	1.7	3.2	4.9	7.0	10.5
0.2	2.1	0.0	0.4	1.0	1.8	2.9	2.1	0.0	0.3	0.9	1.7	3.2
0.4	4.0	0.4	0.0	0.2	0.5	1.2	4.2	0.3	0.0	0.1	0.5	1.4
0.6	5.8	1.0	0.2	0.0	0.1	0.5	5.9	0.9	0.1	0.0	0.1	0.6
0.8	7.7	1.8	0.5	0.1	0.0	0.2	7.5	1.6	0.5	0.1	0.0	0.2
1.0	10.0	2.9	1.2	0.5	0.2	0.0	9.1	2.3	0.9	0.4	0.1	0.0

highly correlated estimates involve a greater proportion of the total information available, and it would appear to be generally true of all least-squares estimates that they can be combined without much loss of information provided there is not an appreciable change in the interaction between the factors involved in each estimate. For example, the direct combination of estimates obtained from the least-squares matrices

$$\begin{bmatrix} 1 & 1-\epsilon & 0 & 0 \\ 1-\epsilon & 1 & 0 & 0 \\ 0 & 0 & 1 & 1-\epsilon \\ 0 & 0 & 1-\epsilon & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1-\epsilon & 0 \\ 0 & 1 & 0 & 1-\epsilon \\ 1-\epsilon & 0 & 1 & 0 \\ 0 & 1-\epsilon & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & 1-\epsilon \\ 0 & 1 & 1-\epsilon & 0 \\ 0 & 1-\epsilon & 1 & 0 \\ 1-\epsilon & 0 & 0 & 1 \end{bmatrix},$$

where ϵ is small, would involve almost complete loss of information since the high correlations may be used to offset each other.

Thus, while the direct combination of least-squares estimates is feasible, it requires a careful consideration of whether the individual groups of observations may augment one another, and here again the possibility of using a transformation presents itself.

6. SURVEY PLANNING

It has been suggested above that, from the point of view of the statistical analyses, orthogonality or partial orthogonality is a property to be desired in survey planning. These suggestions are worthy of further discussion in relation to the execution of the survey and the application of its results. Naturally, if we wish to investigate particular points, emphasis should be laid upon these points to ensure that accurate comparisons can be made. For example, if it is desired to estimate the effect of size of family on individual consumption, a higher proportion of large families than would normally occur might be included in the survey to ensure that this comparison is as accurate as possible. Again, the joint and independent estimation of two or more effects requires the use of orthogonal samples, and the existence of interactions is best recognised with orthogonal samples. However, apart from the difficulties of collecting such a sample, other difficulties will often make it impracticable. Over-all comparisons will normally be made by weighting estimates from different groups according to the proportions in some standard population which is used as a basis of comparison, and if this results in a relatively large loss of information then some other method must be used. Thus the process of taking an orthogonal sample might lead to an imbalance in the proportions of families of different sizes or in the proportions of children of different ages. Also, an orthogonal sample will not necessarily measure what is required whereas this might be provided by a nonorthogonal sample. For example, consider an investigation in which we have two economic classifications, "rich" and "poor," and two family size classifications, "large" and "small"; the virtual nonexistence of the "large-rich" groups would necessitate a great deal of extra sampling if the two comparisons are to be made orthogonal. However, in fact the comparison of the dietary practices of large and small

families must inevitably be referred to the "poor" group if the comparison is to be a valid one, and correspondingly in determining the effect of income group small families must be used. Hence the extra sampling works to the disadvantage of the final result.

This rather baldly leaves out of account finer points such as the interaction of the two factors both from the statistical point of view and in consequence of the maxim that a rise in family size causes a relative decrease in income, but it is fairly obvious that orthogonality is not always to be desired. Nevertheless, when the sets of factors involved scarcely interact, if an orthogonal sample can be collected without undue difficulty the subsequent analysis might be greatly simplified. Similarly, if the population gives rise to almost orthogonal samples, then orthogonal samples might be taken and subsequently adjusted for the effect of interactions on the over-all comparisons that are to be made. Larger departures from orthogonality might require, if interaction is present, a partially-orthogonal sampling design.[5] There are a large number of such designs depending upon various partitions of sums of squares which can be carried out for any or all of the methods of classification, so that it will usually be possible to find a design which would be reasonably representative. However, if the method of weighting suggested above is adopted, then the factors whose effects vary with size of family, i.e., economic, geographical, and seasonal effects, must be eliminated first of all. Thus the sample should be designed to allow these factors to be eliminated simultaneously with a minimum of trouble. Only when this can be done will attention to the other factors be justified.

So far nothing has been said about the method of sampling families of different sizes and the best methods of estimating the individual consumption of children of different ages. Theoretically these can be studied by fractional replication designs, but from the practical viewpoint this is both tedious and wasteful in relation to the comparisons that will finally be made. In practice, therefore, we shall with certain reservations usually be content with stratifying for other factors, including possibly size of family, and analysing the results by the analysis of covariance. The reservations arise in consequence of the interactions of family constitution with other factors, which may cause the almost complete identification or confounding of two or more effects if special action is not taken. Thus, for example, special action may be necessary to prevent the effect of size of family being identified with an increase in age and manifested by an apparent decrease in adolescent consumption.

Hence, to summarise, the sample should deviate from a random or stratified random sample only in so far as it is necessary to ensure that

[5] If a sample is taken so that two or more effects are nearly orthogonal, then it is often possible to treat the sample as if it were orthogonal without undue qualms.

particular effects are distinguished and, if possible, to allow economic, geographical, and seasonal effects to be eliminated simultaneously.

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THE THEORY OF DEMAND APPLIED TO THE FRENCH GAS INDUSTRY

By Michael J. Verhulst

Data for the French gas industry is studied under the assumption that the demand for gas by a consumer is a function of the price of gas and of the disposable income of the consumer. The demand equation is shown to be identified within the model defined by the demand equation, the equation of production, and the equilibrium equations. Elasticities of demand with respect to price and disposable income are obtained and confidence regions are calculated and interpreted.

SUMMARY

The object of the article is to study the demand for manufactured gas in the French gas industry in the fourth quarter of 1945. The data available relate to a sample of 46 firms divided into three groups.

Assuming that the demand for gas by a consumer is a function of the price of gas and of the disposable income of the consumer, it is attempted to measure the elasticities of demand for gas with respect to price and disposable income, respectively.

By considering together the demand equation, the equation of production, and the equilibrium equations, it is possible to show that the demand equation is identified within the model defined by these equations, so that the elasticities of demand can be computed.

It is found that the elasticity of demand with respect to the price of gas is about minus three and the elasticity of demand with respect to the disposable income is about zero. Moreover, confidence regions are calculated and their shape suggests that the values of the elasticities may vary widely with the size of the firms constituting the sample analyzed.

* * *

In a previous article[1] we have tried to determine the form of the production function of a gas firm (manufactured gas) in the neighborhood of the point of equilibrium, and we have made use of an economic model. In the model, we have assumed that the revenue function of a gas firm in the neighborhood of the point of equilibrium can be represented by an equation of the form

$$z = bp^{\beta_0}, \tag{1}$$

where z stands for the gas receipts of the firm, p stands for the gas output of the firm, and b and β_0 are two constants.

[1] "The Pure Theory of Production Applied to the French Gas Industry," Econometrica, Vol. 16, October, 1948, pp. 295–308.

With the statistical data available at the time of writing the former article, it was not possible to determine the value of β_0. Some supplementary data now available make possible this determination. The object of the present article is to study the demand in the French gas industry in the fourth quarter of 1945. But now we take into account the disposable income of the consumers.

In economic theory, it is shown that the demand for a commodity can be considered as a function of the price of the commodity, the prices of other commodities, and the disposable income of the consumer. However, in a concrete study, the statistical data are not such that the determination of this function is immediately possible. It is necessary to make restrictive assumptions.

We shall assume that the demand for gas by a consumer is a function of the price of gas and of the disposable income of the consumer and independent of the variations of the prices of all the other commodities. Let q be the quantity per capita of gas that the individual stands ready to consume, p the price of gas, r the disposable income per capita. We assume that the demand function can be represented by an equation of the form

$$q = kr^{\alpha}p^{\beta}, \tag{2}$$

where k, α, and β are constants. We notice that β and α are the elasticities of demand for gas with respect to the price and the disposable income, respectively.

It is known in mathematics that any continuous function of two variables can be represented in the neighborhood of a point by an equation of the type (2), where α and β are constants. To be able to know whether or not the type of function adopted is admissible, the ideal method would be to conduct an experiment, imposing alternative prices and levels of income on the consumer and studying his reactions. But such an experiment is not possible. However, the statistical data available make the determination of the function (2) possible in an indirect way, by means of econometric methods. But before laying down these methods, it is necessary to present the statistical data available.

For each of the 46 firms listed in the table for Groups I, II, and III, we know the number of consumers supplied, the index of the average disposable income of a consumer, the average price of the gas consumed, and the average quantity of gas consumed per capita. These data are obtained from the results of an inquiry made by the Office Professionnel du Gaz in Paris at the beginning of 1946. They relate to the operation of 46 gas firms in the fourth quarter of 1945.[2]

[2] The figures for numbers of consumers are those of 1946.

The figures for quantity consumed and for price need no commentary. But a word of explanation is necessary for the figures for indices of disposable income. It is not possible to obtain the figures for disposable income in a direct way, so we have established indices of disposable income in an indirect way as follows:

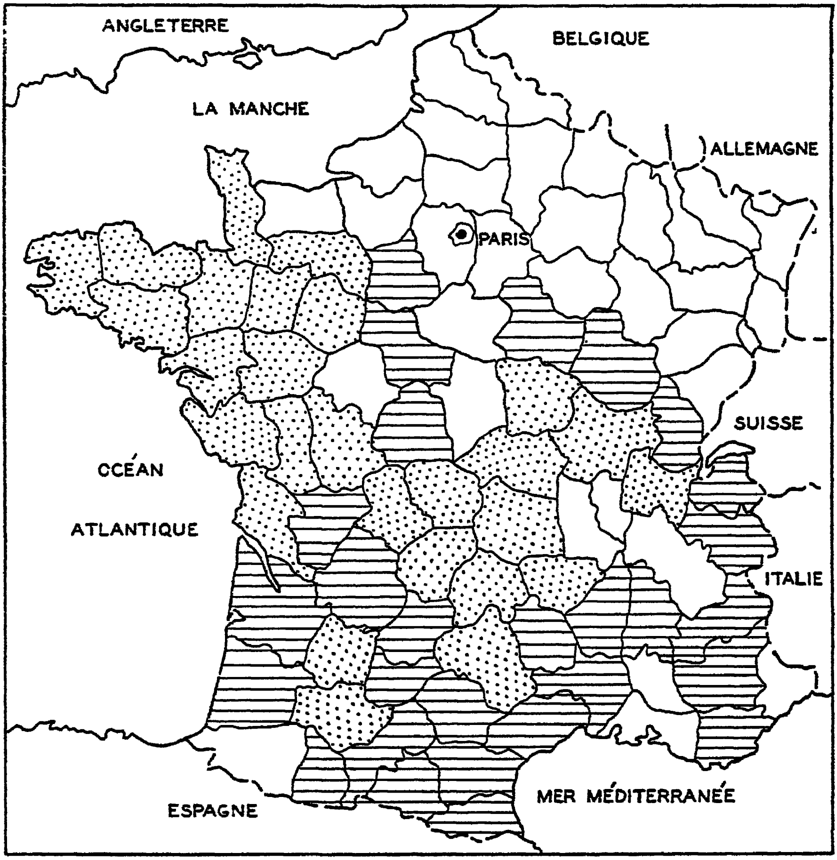

FIGURE 1.—The map of France is divided into three regions. In region I (white) the people are mostly engaged in industrial activity. In regions II and III the people are mostly engaged in agricultural activity. However, in region II (shaded) there is less than one head of cattle for three hectares of land, and in region III (dotted) there is more than one head of cattle for three hectares of land.

The provinces of France have been divided into three groups (see the map of France, Figure 1), as was done in a study made by E. Genissieu[3]

[3] E. Genissieu, "Influence du tarif sur la consommation, la recette, et le bénéfice dans la distribution de l'énergie électrique," *Revue Générale de l'Electricité*, Vol. 35, April, 1934.

TABLE I—FRENCH GAS PRODUCERS

Group and Location	Number of Consumers		Index of Income		Price (francs per m³)		Quantity (cubic meters)	
	N	log N	r	log r	p	log p	q	log q
Group I:								
Rueil (Seine et Oise)	78,377	4.894	25,200	4.401	3.62	0.559	171	2.233
Orléans (Loiret)	22,423	4.351	"	"	4.65	0.667	95	1.978
Châlons sur Marne (Marne)	6,027	3.780	"	"	5.27	0.722	87	1.940
Compiègne (Oise)	5,874	3.769	"	"	3.99	0.601	110	2.041
Lunéville (Meurthe et Moselle)	5,499	3.730	18,900	4.276	3.79	0.579	64	1.806
Meaux (Seine et Marne)	5,415	3.734	"	"	4.60	0.663	114	2.057
Fécamp (Seine Inférieure)	4,003	3.602	"	"	5.00	0.699	77	1.886
Soissons (Aisne)	3,846	3.585	"	"	5.55	0.744	74	1.869
Bayeux (Calvados)	1,836	3.264	25,500	4.407	4.55	0.658	112	2.049
Crèpy-en-Valois (Oise)	1,127	3.052	"	"	4.12	0.615	62	1.792
Commercy (Meuse)	805	2.906	"	"	4.49	0.652	85	1.929
Meung-sur-Loire (Loiret)	698	2.844	23,000	4.362	6.24	0.795	39	1.591
Gisors (Eure)	672	2.827	"	"	5.72	0.757	82	1.914
Joinville (Haute Marne)	610	2.785	"	"	5.59	0.747	42	1.623
Pont-Ste.-Maxence (Oise)	518	2.714	"	"	4.83	0.684	91	1.959
Ste. Menehould (Marne)	472	2.674	19,900	4.299	5.21	0.717	44	1.643
Bar-sur-Seine (Aube)	442	2.645	"	"	4.76	0.678	63	1.799
La Capelle (Aisne)	196	2.292	"	"	4.48	0.651	102	2.009
Group II:								
Nîmes (Gard)	18,778	4.273	180	2.255	5.02	0.701	90	1.954
Annecy (Haute Savoie)	6,857	3.836	221	2.344	3.45	0.538	74	1.869
Chartres (Eure et Loir)	6,720	3.827	196	2.292	4.12	0.615	112	2.049
Auxerre (Yonne)	5,189	3.715	226	2.354	4.52	0.655	66	1.819
Bergerac (Dordogne)	3,443	3.537	152	2.182	5.38	0.731	59	1.771

TABLE I—(continued)

Group and Location	Number of Consumers		Index of Income		Price (francs per m³)		Quantity (cubic meters)	
	N	log N	r	log r	p	log p	q	log q
Group II (continued):								
Beaune (Côte d'Or)	2,633	3.420	212	2.326	4.46	0.649	62	1.792
Draguignan (Var)	2,454	3.390	179	2.253	4.63	0.666	67	1.826
Argenton-sur-Creuse (Indre)	2,256	3.353	264	2.422	6.16	0.790	54	1.732
Tonnerre (Yonne)	903	2.956	203	2.307	5.04	0.702	39	1.591
Mer (Loir et Cher)	682	2.834	248	2.394	5.57	0.746	31	1.491
Montoire (Loir et Cher)	581	2.764	212	2.326	5.22	0.718	51	1.708
La Rochefoucauld (Charente)	569	2.755	168	2.225	5.03	0.702	63	1.799
Sainte Maxime (Var)	419	2.622	127	2.104	5.20	0.716	71	1.851
Seurre (Côte d'Or)	399	2.601	182	2.260	5.78	0.762	47	1.672
Group III:								
Nantes (Loire Inférieure)	58,177	4.765	225	2.352	4.08	0.611	97	1.987
Angers (Maine et Loire)	27,542	4.440	262	2.418	4.16	0.619	97	1.987
Rennes (Ille et Villaine)	27,241	4.435	248	2.394	3.70	0.568	114	2.057
Nevers (Nièvre)	7,210	3.858	206	2.314	4.08	0.611	102	2.009
Chôlet (Maine et Loire)	6,092	3.785	233	2.367	4.64	0.667	81	1.908
Villeneuve-sur-Lot (Lot et Garonne)	1,836	3.264	104	2.017	5.14	0.711	84	1.924
Cosne (Nièvre)	1,545	3.189	175	2.243	5.23	0.718	59	1.771
Villefranche de Rouergue (Aveyron)	1,428	3.155	154	2.187	5.50	0.740	57	1.756
Redon (Ille et Villaine)	1,348	3.130	141	2.149	5.58	0.747	69	1.839
Bressuire (Deux Sèvres)	1,195	3.077	193	2.286	4.91	0.691	58	1.763
La Ferté Bernard (Sarthe)	850	2.929	149	2.173	4.66	0.668	83	1.919
Fouras (Charente Maritime)	820	2.914	234	2.369	6.67	0.824	69	1.839
Bonnétable (Sarthe)	515	2.712	204	2.310	4.06	0.608	56	1.748
Murat (Cantal)	426	2.629	159	2.201	5.49	0.740	58	1.763

on the demand for electricity in France. In the first group, we have the provinces where, in the towns of less than 80,000 inhabitants, the people occupied in agricultural activities are the minority of the working population. In other words, the provinces of the first group are mainly industrial. In the second and third groups, we have the provinces where, in the towns of less than 80,000 inhabitants, the people occupied in agricultural activities are the majority of the working population. But in the second group, there is less than one head of cattle for three hectares of land, and in the third group more than one head of cattle for three hectares of land.

We immediately see that this classification has some other advantages. It takes into account the differences in climate and it corresponds to the differences in the facilities of coal supply since the industrial areas are concentrated around the mines. It therefore seems that a division of the country on these bases is acceptable with regard to the similarity of consumers' preferences and habits.

To obtain the indices of disposable income of an average consumer, we have made use of two series of figures. For the industrial region (Group I) we use the figures of the average wages of the workers in the gas firms. However, as the dispersion of the figures is too great to correspond to the probable dispersion of the true incomes, we have divided the firms into subgroups of three or four firms according to the number of consumers supplied, and computed the average wages of the workers in the subgroups. It is these averages that are taken as indices of the average incomes of the consumers.

In the agricultural regions we have proceeded otherwise. It is possible to obtain the figures for the number of inhabitants in the areas supplied by the gas firms.[4] As we also know the figures for the number of consumers,[5] it seems that the ratio of number of consumers to number of inhabitants is a good index of the disposable income of the consumers. In fact, for agricultural homogeneous regions—homogeneous from the point of view of the similarity of economic preferences and habits—the main economic reason that explains that this ratio is not a constant is that the average disposable income of the groups of consumers is different. These ratios, therefore, multiplied by 1,000, i.e., the number of consumers for 1,000 inhabitants, have been taken as indices of disposable income in the agricultural regions.

Turning now to the statistical methods used,[6] it is enough to say that

[4] Figures for 1946.

[5] The figures for the number of inhabitants and the number of consumers have been supplied by the *Direction du Gaz* of the *Ministère de la Production* in Paris.

[6] In what follows I had the help of Mr. Herman Rubin, to whom I am very much indebted.

these methods are those used by Messrs. M. A. Girshick and Trygve Haavelmo in their recent paper on the statistical analysis of the demand for food.[7]

In the case of the gas industry, it is possible to show that the equation (2) taken together with the three equations considered in the article on the production aspect[8] is identified within the model defined by the four equations. The determination of the coefficients of equation (2) is then as follows:

Denoting by z_1, z_2, the logarithms of the disposable income per capita and of the number of consumers, respectively, and by y_1, y_2, the logarithms of the quantity per capita of gas consumed and of the price of gas, respectively, we compute the moments m_{zy} relative to each group of firms. The results are shown in Table II.

TABLE II—MOMENTS OF OBSERVATIONS

GROUP I

	z_1	z_2	y_1	y_2
z_1	0.9071	2.3350	0.7069	−0.1784
z_2		139.2595	19.4850	−6.1290
y_1			8.4860	−1.9574
y_2				1.2416

GROUP II

	z_1	z_2	y_1	y_2
z_1	1.286	1.574	0.554	−0.046
z_2		50.751	8.837	−3.330
y_1			3.615	−0.906
y_2				0.756

GROUP III

	z_1	z_2	y_1	y_2
z_1	2.280	7.900	0.702	−0.633
z_2		86.280	11.010	−5.860
y_1			2.040	−0.857
y_2				0.926

From these moments we compute first the following expression

$$\begin{pmatrix} m_{z_1z_1} & m_{z_1z_2} \\ m_{z_2z_1} & m_{z_2z_2} \end{pmatrix}^{-1}.$$

[7] M. A. Girshick and Trygve Haavelmo, "Statistical Analysis of the Demand for Food: Examples of Simultaneous Estimation of Structural Equations," ECONOMETRICA, Vol. 15, April, 1947, pp. 79–110.

[8] See the article cited above. We give in the Appendix the three equations dealt with in the article.

Then we compute

$$p_{yz} = \begin{pmatrix} m_{y_1z_1} & m_{y_1z_2} \\ m_{y_2z_1} & m_{y_2z_2} \end{pmatrix} \begin{pmatrix} m_{z_1z_1} & m_{z_1z_2} \\ m_{z_2z_1} & m_{z_2z_2} \end{pmatrix}^{-1},$$

which leads to an expression of the form

$$P_{yz} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

We then compute the values of β and α by the formulae

$$\beta = \frac{p_{12}}{p_{22}}$$

and

$$\alpha = p_{11} - p_{21}\beta.$$

The results are as follows for the three groups:[9]

	Price elasticity β	Income elasticity α
Group I	−3.115	0.166
Group II	−2.492	0.341
Group III	−2.339	−0.341

The values of β are consistent between groups but the values of α are not. This follows from the imperfection of the series of indices of disposable income adopted. However, the figures of disposable income have but a small influence on the values of β, so that we can say that within the model adopted the values of β are good approximations of the elasticity of demand for gas with respect to the price. The dispersion of the values of α seems to show that the demand for gas is independent of the disposable income of the consumers in the neighborhood of the point of equilibrium. However, in view of the inadequacy of the disposable-income series, the latter statement is made with due reserve.

We have calculated *confidence regions* for β and for α and β jointly in the case of Group I. For the other groups the corresponding regions would have a similar shape.[10]

[9] We have computed also the values of β and α corresponding to Groups II and III taken together when the figures of disposable income adopted are the average wages of the workers in the gas firms, as has been done in Group I. These values are $\beta = -3.049$ and $\alpha = -0.447$.

[10] See T. W. Anderson and H. Rubin, "Estimation of the Parameters of a Single Equation in a Complete System of Stochastic Equations," *Annals of Mathematical Statistics*, Vol. 20, March, 1949, pp. 46–63.

For β alone we obtain:

5%	$\beta < -0.729$ or $\beta > 17.58$,
1%	all values of β.

For α and β jointly, the results are indicated in Figure 2. The confidence regions are situated inside the hyperbolae drawn on the graph.

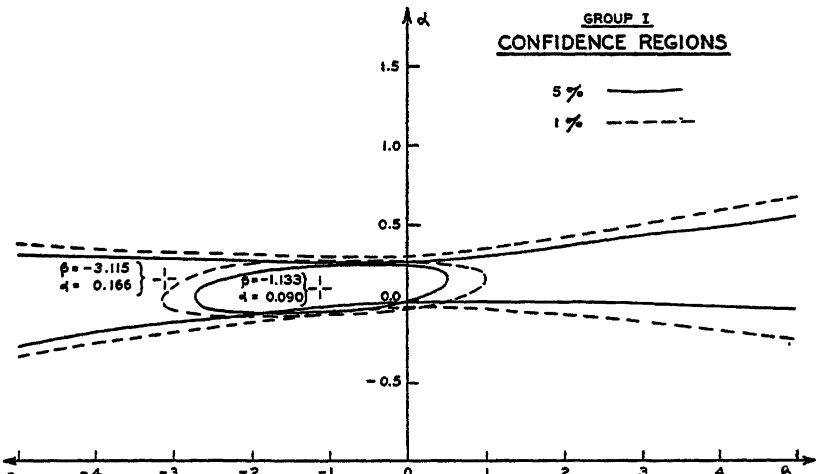

FIGURE 2.—This shows the confidence regions for α and β jointly in the case of Group I (gas firms situated in industrial provinces). The confidence regions inside the hyperbolae are obtained by means of the new statistical methods, and those inside the ellipses by means of an ordinary correlation analysis.

We can compute also the values of β_0 corresponding to equation (1). We first transform equation (2) into an equation of the type (1) with z, ρ, and r as variables. We have

$$q = kr^{\alpha}p^{\beta};$$

but

$$z = Nqp$$

and

$$\rho = Nq,$$

N being the number of consumers. Then

$$\frac{\rho}{N} = kr^{\alpha}\left(\frac{z}{\rho}\right)^{\beta},$$

i.e.,

$$z = (kN)^{-1/\beta} r^{-\alpha/\beta} \rho^{1+1/\beta};$$

and we see that

$$\beta_0 = 1 + \frac{1}{\beta}. \tag{3}$$

The values of β_0 are for the three groups:[11]

Group I: $\beta_0 = 0.679$,
Group II: $\beta_0 = 0.598$,
Group III: $\beta_0 = 0.572$.

In addition to the preceding computations, we thought that it would be useful to compare the results obtained by means of the new statistical methods with the results which can be obtained by means of an ordinary correlation analysis (least-squares method). The results are as follows:

	Price elasticity β	Income elasticity α	β_0
Group I	−1.133	0.090	0.117
Group II	−0.606	0.134	−0.650
Group III	−0.206	0.114	−3.85

The least-squares confidence regions for β are in the case of Group 1:

5% $\quad -2.417 < \beta < -0.049$,

1% $\quad -2.908 < \beta < 0.042$,

and for α and β jointly the results are indicated in Figure 2. The confidence regions are situated inside the ellipses drawn on the graph.

In conclusion, it seems that we can say that, in the neighborhood of the point of equilibrium, the elasticity of demand for gas with respect to the price of gas is about −3, and the elasticity of demand for gas with respect to the disposable income is about zero. However, the latter statement is made with due reserve, as has been explained above.

Further, the shape of the confidence regions suggests that the value of β may vary widely according to the size of the firms constituting the sample analyzed.

It is important also to emphasize the difference between the results obtained by means of the new statistical methods and the results obtained by means of an ordinary correlation analysis. It is now well known that the second method is wrong.[12]

[11] And in the case considered in the footnote: $\beta_0 = 0.672$.

[12] See T. Koopmans, "Statistical Estimation of Simultaneous Economic Relations," *Journal of the American Statistical Association*, Vol. 40, December, 1945, pp. 448–466.

Finally, as in the article on the production aspect of the French gas industry, we found that the values of $\alpha'\beta_0$ and $\alpha''\beta_0$ are on the average:

$$\alpha'\beta_0 = 0.82, \qquad \alpha''\beta_0 = 0.12,$$

where α' and α'' are the elasticities of gas output with respect to costs and capital charges respectively (see the Appendix); we see that the values of α' and α'' are on the average:

$$\alpha' = 1.22, \qquad \alpha'' = 0.18.$$

If we assume that there is perfect competition in the markets of the factors of production, the value of ϵ defined by

$$\epsilon = \alpha' + \alpha'' \tag{4}$$

shows that there are increasing returns to scale.

APPENDIX

The equations of each structure of the model have the following form as far as the production aspect is concerned:

$$\begin{aligned} Z - \alpha'\beta_0 X - \alpha''\beta_0 Y &= \kappa_1 + \epsilon_1 , \\ Z - X &= \kappa_2 + \epsilon_2 , \\ Z - Y &= \kappa_3 + \epsilon_3 , \end{aligned}$$

where Z stands for the logarithm of z, z being the gas receipts of the firm; where X stands for the logarithm of x, x being the net prime costs of the firm, i.e., prime costs minus by-products receipts; where Y stands for the logarithm of y, y being the capital charges of the firm; and where α', α'', β_0 , κ_1 , κ_2 , κ_3 are constants within each structure, but where ϵ_1 , ϵ_2 , ϵ_3 are random variables independently distributed by producer, with means zero and constant covariance matrix.

The first of these three equations is the production function, and the two others are the profit-maximization equations.

α' and α'', of course, are the elasticities of output with respect to costs and capital charges respectively, and β_0 is defined by relation (3), where β is the elasticity of demand for gas with respect to price.

International Bank for Reconstruction and Development

A NOTE ON THE PURE THEORY OF PRODUCTION

By Kenneth May

The pure theory of production, formulated in terms of demand and production functions, is a classic part of partial equilibrium theory. It is commendable that efforts are being make to link this theory with cost, output, and price data. Perhaps the most difficult phase of such work is the matching of economic situations with appropriate models chosen from the variety available in the pure theory. Since statistical analysis assumes the appropriateness of a model and is concerned with estimation of its parameters, an inappropriate model may still lead to deceptively satisfying empirical results. It is in connection with this problem that we comment upon the economic hypotheses of a recent article.[1]

The paper deals with twenty-five firms producing manufactured gas in France in the last quarter of 1945. The economic hypothesis is the classic monopoly model. Each firm is assumed to have a production function involving two factors and a revenue function involving the amount of gas produced and sold by the firm. The revenue function is equivalent to and could be replaced in the model by a demand function by means of the identity: revenue equals price times output.[2] The classic equilibrium conditions are derived by maximizing the profit of each firm on the assumption that it can set its output (or price) at will, the price (or output) then being determined by the demand function (or revenue function). Output, price, and factor amounts are then determined by the familiar marginal conditions. The paper is, of course, concerned mainly with estimating the elasticities involved in the model.

Do the economic hypotheses underlying this model apply to the French gas industry in the last quarter of 1945? For many years the French gas industry had belonged to that category of public utility which was "run with private capital and municipal participation and control."[3] The municipality had considerable control over the manage-

[1] Michel J. J. Verhulst, "The Pure Theory of Production Applied to the French Gas Industry," Econometrica, Vol. 16, October, 1948, pp. 295–308. The same model with a modified demand function is used by the author also in "The Theory of Demand Applied to the French Gas Industry," Econometrica, Vol. 18, January, 1950, pp. 45–55. The present note is the outcome of correspondence between Verhulst, J. Marschak, and the writer on several interesting questions raised by these papers. We are concerned here exclusively with the appropriateness of the model used.

[2] In the second article cited in the previous footnote, Verhulst uses a demand function involving consumer income and points out the relation between the revenue and demand functions.

[3] O. C. Hormell, *Control of Public Utilities Abroad*, Syracuse, New York: Syracuse University, 1930, p. 21. (Reprint, originally published as an appendix

ment of the enterprise. Often a minimum return was guaranteed to the shareholders, the city making up any resulting losses and sharing in profits when they occurred. Rates, conditions of service, and numerous other details were subject to control by the municipality and, in the last resort, by the Prefect and the Ministry of Public Works. Rates were fixed, not on the basis of maximum profit, but in terms of cost and public need. Similarly, the producer was required to give specified quantity and quality of service. In short, the industry was in a position somewhat similar to that of the gas industry in the United States, but if anything less free to make price and output adjustments.[4]

In the last quarter of 1945, the industry was shifting toward nationalization, which was to take place in April, 1946. "At that time, the municipalities had lost control over rates to the 'Comité National des Prix' in Paris, under the direction of the Ministry of National Economy. Rates were related to those existing at the beginning of the war by means of indices compiled according to the costs of materials entering into the manufacture of gas. From time to time, higher rates would be decided upon when justified by the increase of costs as evidenced in the increasing number of firms in deficit."[5]

In such an environment, what is the optimum problem for the firm? The price is set by public authority. Hence it cannot choose the price and output which yield maximum profit. If it ignores the demand function, it might set its output in order to maximize profit for the fixed price.[6] But the firm is well aware that such a procedure is not feasible. It is faced not only with a fixed price but with the obligation to give a certain quantity and quality of service. Its optimization problem, as

to the Report of the Commission on the Revision of the Public Service Commissions Law of the State of New York.)

[4] *Ibid.*, pp. 21–41. The public status of price is emphasized by the following quotation from page 29: "Under the French law the charge to the consumer or user of the services of a public utility is considered to be a special *impost*. An impost can be laid only under authority of parliament. Regulation of rates of a public utility remains a public affair notwithstanding any features of the contractual side of a concession. Public service rates are not a table of prices fixed by contract. In practice rates may be proposed by the concessionaire, or established by the public authority upon the recommendations of the concessionaire, but never can the rates become effective without the approval of the public authority designated by law. In law, therefore, the fixing of rates is an act of regulation performed by a competent public authority. Hence, a rate may be modified at any moment without the consent of the concessionaire or the consumer, notwithstanding even the contract which the consumers may have with the concessionaire, unless otherwise provided by law."

[5] Verhulst, in correspondence with the writer.

[6] See G. C. Evans, *Mathematical Introduction to Economics*, New York: McGraw-Hill, 1930, pp. 8–9.

far as economic action is concerned, is then exclusively one of internal efficiency, the familiar minimization of the cost of producing a given output.

At the same time the firm will try by political means to secure favorable conditions from the regulating authority, but it can hardly hope to persuade the government to fix prices at the level which gives maximum monopoly profit. Government regulation of utilities has been established precisely to force monopolies to higher outputs and lower prices than would prevail under unregulated conditions. Nor is there any reason to think that the fixed price and corresponding output will be at all near the monopoly values. Obviously the demand for gas is such that an unregulated monopoly could increase its price considerably before reaching the maximum profit, and it is just this possibility which impels public regulation.[7]

The economic hypotheses of the paper under consideration lead to the equilibrium conditions $\alpha'\beta_0 = x/z$ and $\alpha''\beta_0 = y/z$, where x, y, and z are respectively "net prime cost," "an index of capital charges," and revenue, and where α', α'', and β_0 are the elasticities of output with respect to x and y and of demand. Estimates are found for $\alpha_1' = \alpha'\beta_0$ and $\alpha_2' = \alpha''\beta_0$. We might expect that these estimates would correspond to some extent with the ratios x/z and y/z of the data if the model were appropriate. The estimates of these parameters for the ten larger firms are $\alpha_1' = 0.83$ and $\alpha_2' = 0.10$ and for the fifteen smaller firms, $\alpha_1' = 0.80$ and $\alpha_2' = 0.14$. Using the data given in the paper, we find that for the ten larger firms x/z ranges from 0.75 to 1.51 with an average of 0.93. The estimate of α_1' is almost at the lower end of the range of values of x/z. For the fifteen smaller firms x/z varies from 0.804 to 1.88 with an average of 1.27. The estimate of α_1' is 0.80, which is smaller than any of the x/z for this group. The values of y/z from the data for the larger firms vary from 0.131 to 0.258, in contrast to an estimate of α_2' at 0.10. For the smaller firms y/z ranges from 0.098 to 0.85 with only two out of seven values below 0.39, yet the estimate of α_2' is 0.14.[8] The values of x/z and y/z seem to be consistently larger than we would expect from the theory and the estimates of the parameters.

These discrepancies suggest the possibility that the economic hypothesis is inappropriate, but they might well be accounted for by difficulties

[7] In the case worked out by Evans and cited in the previous footnote, the monopoly sets its output so as to maximize profit for the fixed price; the regulating authority sets the price so as to equalize demand and supply. The resulting price is substantially lower and the output higher than for the case of unrestricted monopoly. Such a price-fixing model would be more realistic than that used in the paper under discussion, but for the reasons cited a regulated monopoly is actually faced with given price and output.

[8] Values of y are given in the data for only half the firms.

associated with aggregation, faulty data, etc.[9] The decisive point is that the institutional framework of the French gas industry in the period considered did not permit the realization even approximately of the assumptions of the classic monopoly model and in fact was designed specifically to achieve price and output substantially different from those given by the well-known monopoly equilibrium conditions.

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[9] Verhulst writes: "The reasons for the discrepancy between the values of α_1', α_2', and x/z are, in my opinion, first of all that the model does not consider the output of by-products produced as a supplementary variable, and correspondingly considers the *net* prime cost instead of the prime cost, and second that the index of capital charges is a bad index especially for the smaller firms. It is not possible to say anything about the values of y/z because the values of y are not the actual values of the capital charges; they are an index of these charges. The discrepancy shows only that the values of α_1' should normally be slightly higher. Moreover, the coefficient which really matters is the sum $\alpha_1' + \alpha_2'$ equal to 0.93 for the larger firms and 0.94 for the smaller."

HOMOGENEOUS SYSTEMS IN MATHEMATICAL ECONOMICS: A COMMENT[1]

By Kenneth J. Arrow

In a recent paper[2] in this journal, Professor Tintner has developed some highly useful general theorems bringing together the various problems in economic theory which lead to relations homogeneous of degree zero in certain variables. These theorems were derived by Professor Tintner under the assumption that the various functions involved are differentiable twice.[3] Now, there appears to be a rapidly developing tendency in modern mathematical economics to avoid assumptions of differentiability which seem to have no economic relevance and which frequently are unnecessary. Indeed, the use of methods of the differential calculus occasionally leads to unnecessarily complicated derivations and exposition.[4] It is the purpose of the present note to show that Professor Tintner's chief results can be obtained without any assumptions of differentiability and in a simpler manner. It will consequently be noted that some remarks of Professor Tintner implying that differentiability is a necessary condition for homogeneity are misleading.

* * *

The tools used by Professor Tintner are the following two theorems. Let the behavior of some individual be described by saying that he maximizes $g(x, x^*, x^{**})$ with respect to x and x^* subject to the restrictions, $h^{(k)}(x, x^*, x^{**}) = 0$, $(k = 1, 2, \cdots, N)$. Here x, x^*, x^{**} are vectors; let $x_1, \cdots, x_p$ be the components of x.

Theorem 5: *If the function g which is to be maximized and the side conditions $h^{(k)}$ are all either independent of the variables that are components of x^{**} or homogeneous of some arbitrary degrees in the same variables,*

[1] I wish to thank J. Marschak, Cowles Commission for Research in Economics and The University of Chicago, and G. Tintner, Iowa State College, for their very helpful comments and criticism.

[2] G. Tintner, "Homogeneous Systems in Mathematical Economics," Econometrica, Vol. 16, October, 1948, pp. 273–294.

[3] *Ibid.*, p. 277.

[4] This point has been especially stressed by Samuelson. See P. A. Samuelson, *Foundations of Economic Analysis*, Cambridge, Massachusetts: Harvard University Press, 1947, pp. 70–76, 107–112; and "Comparative Statics and the Logic of Economic Maximizing," *Review of Economic Studies*, Vol. 14 (1), 1946–47, pp. 41–43. It is true, however, as pointed out by one of the reviewers of Samuelson's book, that the differentiability or nondifferentiability of the utility or revenue function or of the constraints on behavior is of importance in characterizing the behavior of the decision variables with respect to variations in the initial conditions. See K. E. Boulding, "Samuelson's *Foundations*: The Role of Mathematics in Economics," *Journal of Political Economy*, Vol. 56, June, 1949, p. 194.

then the solutions $x_1, \cdots, x_p$ *are homogeneous of zero degree in these variables.*

THEOREM 6: *If the function* g *which is to be maximized and the side conditions* $h^{(k)}$ *are either independent of the variables that are components of* x^* *and* x^{**} *or homogeneous of some arbitrary degrees in the same variables, then the solutions* $x_1, x_2, \cdots, x_p$ *are homogeneous of zero degree in the variables that are components of* x^{**}.

Following Professor Tintner, a function $f(u, u^*)$ of two vectors u, u^* is said to be homogeneous of degree K in u if $f(tu, u^*) = t^K f(u, u^*)$ for every *positive* value of t.[5] It may be observed that if f is independent of u, it is homogeneous of degree zero in u.

Theorem 5 will first be derived from Theorem 6.

Proof of Theorem 5: Let $x' = (x, x^*) = (x_1, \cdots, x_r)$, and let x'' be a vector whose components are variables not appearing in g or in any of the functions $h^{(k)}$. Under the assumption of Theorem 5, g and $h^{(k)}$ are each homogeneous of some degree in x'', x^{**}, or independent of them. Then, Theorem 6 applies, with x', x'' replacing x, x^{**}, respectively, so that the solutions $x_1, \cdots, x_r$, and, in particular $x_1, \cdots, x_p$, are homogeneous of degree zero in x^{**}.

Proof of Theorem 6: Let $\hat{x}$, $\hat{x}^*$ maximize g subject to $h^{(k)} = 0$ for a given x^{**}. Then, by definition of maximum,

$$g(\hat{x}, \hat{x}^*, x^{**}) \geqq g(x, x^*, x^{**}) \tag{1}$$

for all x, x^* such that

$$h^{(k)}(x, x^*, x^{**}) = 0. \tag{2}$$

Take any $t > 0$. Assume g homogeneous of degree K_0 in x^*, x^{**}, $h^{(k)}$ homogeneous of degree K_k in x^*, x^{**}. Multiply through both sides of (1) by $t^{K_0} > 0$. By the definition of homogeneity, it follows easily that

$$g(\hat{x}, t\hat{x}^*, tx^{**}) \geqq g(x, tx^*, tx^{**}) \tag{3}$$

for all x, x^* satisfying (2). Multiply through in (2) by t^{K_k}; by the definition of homogeneity, (2) is equivalent to

$$h^{(k)}(x, tx^*, tx^{**}) = 0. \tag{4}$$

Therefore, (3) holds for all x, x^* satisfying (4). Replace x^* by (x^*/t) in (3) and (4); this is permissible since $t \neq 0$. We then have

$$g(\hat{x}, t\hat{x}^*, tx^{**}) \geqq g(x, x^*, tx^{**}) \tag{5}$$

[5] Tintner, *op. cit.*, p. 274. The term "positive homogeneity" is frequently used in the above sense, "homogeneity" being reserved for the case where $f(tu, u^*) = t^K f(u, u^*)$ for *all* t.

for all x, x^* satisfying

$$h^{(k)}(x, x^*, tx^{**}) = 0. \tag{6}$$

Clearly, $\hat{x}$, $t\hat{x}^*$ satisfy (6) so that from (5) and (6) and the definition of a constrained maximum, $\hat{x}$, $t\hat{x}^*$ maximize $g(x, x^*, tx^{**})$ subject to $h^{(k)}(x, x^*, tx^{**}) = 0$. Since $\hat{x}$ is unchanged by multiplying all the components of x^{**} by a positive constant, the solutions $x_1, \cdots, x_p$ are homogeneous of degree zero in x^{**}.

It has also been proved, incidentally, that the components of the solution x^* are homogeneous of degree one in x^{**}. Thus, in Professor Tintner's discussion of monopoly,[6] it can be shown that the prices charged by a monopolist are homogeneous of degree one in the prices charged on atomistic markets.

Since differentiability has been shown to be irrelevant for the validity of the basic theorems on homogeneous functions, it cannot be, as Professor Tintner states in several places, that lack of differentiability explains lack of homogeneity. In the case of the kinked demand curve in oligopoly,[7] the point seems to be that the established price must be taken as a variable distinct from price as a decision variable; since it is a datum it is to be included, along with the prices on atomistic markets, in the components of x^{**}. The analysis of oligopoly then proceeds as in Professor Tintner's paper, so that output is homogeneous of degree zero in all the components of x^{**}; but of course, output need not be homogeneous of degree zero in the prices on atomistic markets alone.

Similarly, it cannot be that the existence of lumpiness disturbs homogeneity simply because of the lack of differentiability.[8] Mathematically, lumpiness may be represented by letting certain variables assume only integer values. The previous definitions of homogeneity do not have any meaning in this case.

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[6] *Ibid.*, pp. 284–285.
[7] *Ibid.*, p. 287.
[8] *Ibid.*, p. 294.

A NOTE ON TINTNER'S "HOMOGENEOUS SYSTEMS"

BY CECIL G. PHIPPS

IN ECONOMETRICA for October, 1948, Gerhard Tintner has discussed "Homogeneous Systems in Mathematical Economics." His work is in two parts, a mathematical discussion followed by applications to economic problems. The purpose of this note is to comment upon his mathematical proofs.

First, it is to be noted that the development of the mathematical part would have been much simpler if his theorems had been derived directly from the definition of homogeneity. This procedure would have avoided cumbersome matrices and determinants.

Second, the conditions of his Theorem 3 are only sufficient; a simple example will show that they are not necessary. Let

$$f(x, a) = g(x)^{a_2 - a_1},$$

where x and a have the meaning given them by Tintner. Then f is homogeneous of degree zero in the a. Next let $a_1 = c_1 + 5$, $a_2 = c_2 + 5$, and $a_3 = c_3$. After this substitution, $f(x, c)$ will be homogeneous of zero degree in the c although the a were not homogeneous of any degree in the c.

The results of this theorem are applied near the middle of page 292. There he asks the question: What conditions must be imposed upon the method of forming anticipations in order to retain the homogeneity of degree zero of the solutions? The answer is of course that we do not know what conditions are necessary but we do know some that are sufficient. Tintner, however, has another answer. He states in the following paragraph, "In order to preserve homogeneity of zero degree of the solutions, it is necessary . . .," although the conditions are only sufficient.

The discussion of the subsequent problem must be modified accordingly. Formula (12.2) on the next page will hold only under the assumption that we happen to have homogeneity.

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COMPTE-RENDU DU CONGRES DE COLMAR

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LE CONGRÈS Européen de la Societé d'Econométrie s'est tenu à Colmar, France, du 12 au 14 Septembre 1949. M. François Divisia avait la responsibilité de l'organisation du congrès et M. René Roy celle du programme scientifique et de compte-rendu qui suit.

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Théorie mathématique de l'équilibre économique, équations générales de la dynamique, LUIGI AMOROSO, Université de Rome.

L'ESSAI que je désire présenter à Colmar se rapporte à la note qui est annexée au discours que j'ai eu l'honneur de prononcer le 13 novembre 1948 à l'Académie dei Lincei, en commémoration de V. Pareto et qui a été publiée par la même Académie (Quaderno n. 10 dei *Problemi di Scienza e Cultura*) et par le *Giornale degli Economisti* (Novembre-Décembre 1948). J'ai voulu montrer par mon discours, que la *Théorie mathématique de l'équilibre économique* peut être exposée dans une forme simple et élégante à la fois, si, d'une manière tout-à-fait analogue aux

paramètres qui paraissent dans les équations de la mécanique classique dans leur deuxième forme lagrangienne, les coordonnées, qui représentent une configuration générique de chaque sujet, conservent un sens indéterminé et par conséquent peuvent exprimer aussi bien des quantités possédées que des quantités consommées, produites, épargnées. Les équations de l'équilibre économique peuvent alors être contenues en trois lignes: la première se rapportant aux choix individuels, la deuxième aux liaisons que Pareto dit du premier genre, la troisième aux liaisons du deuxième genre. Les liaisons du premier genre dans les expressions parétiennes sont caractérisées pour se rapporter chacune au même sujet, quel que soit l'objet; celles du deuxième genre, au contraire, pour se rapporter chacune au même objet, quel que soit le sujet.

L'équilibre économique, on le sait bien, ne correspond pas à un état de repos, mais à une configuration du mouvement uniforme (stationnaire par rapport au temps). Le problème dynamique, dans la science économique, jaillit tout court lorsqu'on repousse l'hypothèse de l'uniformité. Je me propose d'envisager quelles sont les conséquences de l'élimination de cette hypothèse à l'égard de la théorie contenue dans la Note susdite. J'en tire des équations, qui, en certaines hypothèses, peuvent être considérées comme les équations générales de la dynamique économique.

DISCUSSION: Les hypothèses de continuité, dérivabilité, etc., sont-elles bien nécessaires? Equations différentielles ou équations aux différences? On évoque le débat: error in variables, error in equations. Le problème est posé du raccord de la théorie parétienne avec l'observation statistique (discontinuités, agrégation, etc.) et avec la politique économique.

La causalité dans le théorème du rendement social, André Nataf, Fonds Monétaire International, et Michael J. Verhulst, International Bank for Reconstruction and Development.

L'attention est toujours plus portée, en économie, sur une étude approfondie des modèles économiques proposés. Ce sont sans doute, à notre connaissance, Samuelson et Schumpeter qui ont insisté le plus systématiquement sur ce point, attirant l'attention sur le fait que des modèles économiques de même forme pouvaient s'appliquer à des états qui, dans la vie de tous les jours, sont considérés comme très éloignés les uns des autres. Nous allons, dans ce qui suit, essayer de voir les causes essentielles de ces différences dans le cadre des théories du rendement social.

Rappelons qu'un état est de rendement social maximum lorsqu'on ne peut augmenter la satisfaction d'un individu sans diminuer en même temps celle d'un autre au moins. Les différents états considérés dans

ces théories se rapportent tous à des conditions de structure techniques et psychologiques supposées fixes.

Le problème, jusqu'ici, quel que soit le mode d'exposition et de démonstration adopté, s'est toujours nettement décomposé en deux parties distinctes, sans lien apparent entre elles: (1°) Etablissement des conditions de production permettant de tirer le parti maximum des ressources dans une structure donnée; (2°) Modes de répartition des produits ne causant pas de perte "sèche" de rendement social.

Dans l'étude du premier problème, on arrive à la conclusion (que nous appellerons C_1) que, quelle que soit la structure politique ou économique considérée, les conditions de maximation de la production sont les mêmes. Cependant, il faut bien voir, avec Mr. Allais, qui, à notre connaissance, a le plus clairement mis ce point en relief, que ceci, non seulement suppose que l'on se donne les facteurs de production au début de la période économique considérée, ce qui semble aller de soi, mais aussi à la fin de cette période, ce qui engage lourdement la décision d'ensemble, en dehors de tout critère fourni par le théorème lui-même.

La conclusion C_1 ne prend tout son sens qu'avec ces conditions aux limites. Si l'on songe que, dans la pratique, bien des facteurs de production sont liés, qu'il existe pour eux des limitations naturelles, obligeant à un choix entre leur utilisation, il apparaît vraisemblable qu'avec les mêmes facteurs de production primaires,[1] C_1 n'empêchera pas que le choix des productions à effectuer dépendra de critères extérieurs à ceux que peut fournir la théorie du rendement social et que ces critères ont un caractère social (au sens ordinaire du mot).

C'est alors, nous semble-t-il, que l'on peut mieux voir les relations causales entre les deux points: production et distribution. Le point important n'est pas que la répartition des revenus doit être telle qu'elle assure une répartition optima d'une production supposée elle-même, et à priori, optima. Mais ce qui est important, c'est que la cause ultime de choix entre différents états de production est la possibilité de certaines productions. Nous avons été profondément frappés par l'analogie qui existe entre cette situation et la notion de solution d'un jeu présentée par Von Neuman et Morgenstern. Il nous semble possible d'appliquer au moins l'esprit de leur méthode pour identifier les différents états de rendement social avec les ensembles de solutions d'un jeu, l'état qui prévaut étant celui que les conditions sociales imposent.

Qu'une fois la solution établie, il soit possible de trouver un système de prix compatible avec un cadre économique de concurrence parfaite, ceci ne s'inscrit pas en faux contre le fait qu'il n'y a pas de possibilité

[1] Ce sont les gisements minéraux et les possibilités agricoles, ainsi que les possibilités de population. Ces facteurs sont les seuls sur lesquels on n'ait que très peu d'action.

de solution unique par le seul jeu de cette concurrence parfaite. Par conséquent, la recherche de la solution par l'établissement empirique d'un système de prix, lorsque l'on désire un état final donné, n'a pas de sens dans l'économie statique lorsqu'on n'ajoute pas d'autres causes motrices. Cette recherche ne prend son sens que si l'on fait intervenir des causes dynamiques qui n'ont aucune raison d'amener à un état final donné, et qui remplacent en tant que causes définissant complètement la solution, la donnée des conditions aux limites.

L'on voit alors la différence entre les régimes de concurrence parfaite selon que celle-ci soit privée ou sociale. Dans le deuxième cas, la nature des choix à faire est nettement et même crûment mise en relief. Dans le premier cas, paradoxalement, malgré l'apparente liberté des joueurs, la solution est déterminée absolument à partir des conditions initiales et de la dynamique économique. Une théorie de cette dynamique est indispensable pour apprécier la valeur pratique des solutions économiques offertes par la concurrence parfaite privée.

DISCUSSION: Il convient de distinguer *finalité* et *causalité*. La notion de finalité ne doit pas seulement évoquer une planification intégrale, mais aussi les heurts d'intentions particulières. Une sorte de finalité, mais sociale, se retrouve chez von Neumann, dans la notion de "solution" d'un jeu. En ce qui concerne la théorie du rendement social, on hésite sur la signification des fonctions de production et sur leur importance et l'on se demande si l'optimum parétien peut se définir sans faire intervenir les prix.

Généralisation de l'inégalité de Bienaymé et ses applications économiques, FELICE VINCI, Université de Milan. (Exposé fait par Mr. Guilbaud en l'absence de Mr. Vinci.)

DANS une distribution de fréquences d'un ensemble d'observations désignons par E_i, $(i = 0, 1, 2, \cdots, n)$, les écarts absolus à partir d'une origine quelconque c, rangés par ordre de grandeur; appelons f_i les fréquences relatives, de telle façon que le moment d'ordre r par rapport à c soit:

$$\mu_{c,r} = \sum_{i=0}^{n} E_i^r f_i \, .$$

Si l'on choisit une constante arbitraire λ, inférieure ou égale à l'écart E_s compris entre les extrèmes, on obtient le moment incomplet:

$$_s\mu_{c,r} \geqq \lambda_r \sum_{i=s}^{n} f_i \, .$$

Il faut donner à λ la forme d'un multiple de l'écart moyen $\sigma_{c,r}$:

$$\lambda = k_r \sqrt[r]{\mu_{c,r}} = k_r \sigma_{c,r} \, .$$

Puis, en divisant les deux membres par $\mu_{c,r}$, on aura le moment incomplet relatif:

$$(1) \qquad {}_{s}m_{c,r} = \frac{{}_{s}\mu_{c,r}}{\mu_{c,r}} \geqq k_r^r \sum_{i=s}^{n} f_i ,$$

d'où:

$$(2) \qquad 1 - \frac{1}{k_r^r}\, {}_{s}m_{c,r} \leqq \sum_{i=0}^{s-1} f_i .$$

Admettons que les valeurs observées varient de façon continue et que les écarts absolus rangés par ordre de grandeur prennent les valeurs réelles comprises entre 0 et ω. Au cas où les fréquences de ces écarts seraient constantes, le moment d'ordre r par rapport à c serait donné par la moyenne arithmétique simple des puissances des écarts, c'est-à-dire:

$$\mu_{c,r} = \frac{1}{\omega} \int_0^{\omega} E^r \, dE = \frac{\omega^{r+1}}{\omega^{(r+1)}} .$$

Substituant dans la précédente intégrale à la limite inférieure 0 la constante arbitraire λ comprise entre 0 et ω, on aura encore:

$${}_{\lambda}\mu_{c,r} = \frac{1}{\omega} \int_{\lambda}^{\omega} E^r \, dE = \frac{\omega^{r+1} - \lambda^{r+1}}{\omega\,(r+1)}$$

d'où le moment incomplet relatif:

$$(3) \qquad {}_{\lambda}m_{c,r} = \frac{{}_{\lambda}\mu_{c,r}}{\mu_{c,r}} = 1 - \left(\frac{\lambda}{\omega}\right)^{r+1} .$$

Donnant à λ et à ω la forme d'un multiple de l'écart moyen $\sigma_{c,r}$:

$$\lambda = k_r \sqrt[r]{\mu_{c,r}} = k_r \sigma_{c,r} ,$$

$$\omega = K_r \sqrt[r]{\mu_{c,r}} = K_r \sigma_{c,r} ,$$

on peut écrire:

$$(3 \text{ bis}) \qquad {}_{\lambda}m_{c,r} = 1 - \left(\frac{k_r}{K_r}\right)^{r+1} .$$

Si l'on substitue dans l'inégalité (2) le précédent résultat à ${}_{s}m_{c,r}$, nous pouvons écrire enfin:

$$(4) \qquad 1 - \frac{1}{k_r^r} + \frac{k_r}{K_r^{r+1}} \leqq \sum_{i=0}^{s-1} f_i .$$

On obtient ainsi pour la fréquence des écarts plus petits en valeur absolue que λ une limite inférieure encore plus approchée de la dite

fréquence que la limite obtenue à partir des deux premiers termes du premier membre qui, pour $r = 2$, exprime l'inégalité bien connue de Bienaymé.

Cette substitution est permise quand les fréquences des écarts décroissent à mesure que croît la valeur absolue de ceux-ci, en ce sens que le moment (3) résulte de ce cas plus grand et permet par conséquent de tirer de l'inégalité (2) une limite encore plus basse, laquelle limite est encore abaissée par la continuité des valeurs absolues, parce que le moment incomplet mis à la place du numérateur de l'équation (3) tend à croître dans une proportion plus forte que le dénominateur: il comprend les écarts les plus grands en valeur absolue.

Cette hypothèse—qu'une distribution des fréquences a un seul maximum et que l'on peut, dans ces conditions, lui substituer une distribution décroissante des valeurs absolues des écarts à partir d'une origine convenablement choisie—est communément admise pour les erreurs d'observation et vaut couramment dans d'autres domaines: elle ne restreint pas sensiblement les perspectives ouvertes par les nouveaux résultats qui, de ce fait, prennent une certaine importance quant à leur applications économiques pour lesquelles on a maintes fois constaté que les formes gaussiennes ou quasi gaussiennes sont inadéquates. D'autre part, l'existence d'un multiple maximum défini K_r de l'écart moyen est un fait expérimental que l'on peut établir avec une marge de sécurité écartant toute estimation inexacte et qui, du reste, pour $r = 2$ ne dépasse pas la valeur de 5.

Pour donner un exemple, supposons que dans l'inégalité (4) $r = 2$ et confrontons les limites inférieures des fréquences pour les écarts plus petits en valeur absolue que les multiples k_2 de l'écart moyen en posant $K_2 = \infty$ (hypothèse de Bienaymé) et $K_2 = 5$:

k_2	$K_2 = \infty$	$K_2 = 5$	DIFFÉRENCE
1.5	0.5556	0.5676	0.0120
2.0	0.7500	0.7660	0.0160
2.5	0.8400	0.8600	0.0200
3.0	0.8889	0.9129	0.0240
3.5	0.9184	0.9464	0.0280
4.0	0.9375	0.9695	0.0320
4.5	0.9506	0.9866	0.0360
5.0	0.9600	1.0000	0.0400

Des tables spéciales ont été dressées pour le maniement rapide de l'inégalité (4) attribuant à K_2 les valeurs entières de 2 à 10 et faisant varier k_2 par intervalles d'un décime à commencer par 1.5.

DISCUSSION: **Le gain n'est peut-être pas numériquement considérable, mais il importe de savoir s'il est le plus grand possible avec les hypothèses faites. La proposition du Prof. Vinci doit être replacée dans le cadre général des perfec-**

tionnements de l'inégalité de Bienaymé-Tchebychef, lesquels sont nombreux: la méthode Vinci peut être retenue et généralisée de façon efficace.

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An International Economic System, JACQUES J. POLAK, International Monetary Fund.

THIS paper summarizes some of the econometric work performed by the Research Department of the International Monetary Fund. An attempt is made to construct a multicountry model which would give a satisfactory explanation of the international transmission of fluctuations in economic activity.

In the simplest model we disregard changes in prices. We assume for each country i the following relations (omitting constant terms):

$$x_i = \sigma_i x_w, \tag{1}$$

$$y_i = \mathfrak{M}_i x_i, \tag{2}$$

$$m_i = \mu_i y_i, \tag{3}$$

where the symbols have the following meaning:

(a) *Variables* (all expressed in millions of dollars at constant prices):

x_i indicates volume of exports of i,
y_i indicates real national income of i,
m_i indicates volume of imports of i,
x_w indicates volume of world trade;

(b) *Parameters*:

σ_i indicates marginal propensity of foreign countries to import from i,
$\mathfrak{M}_i$ indicates foreign trade multiplier of i,
μ_i indicates marginal propensity to import of i.

The system of $3n$ equations for n countries indicated above is completed by a definitional equation:

$$x_w = \sum_i m_i . \tag{4}$$

This system represents the mechanism of transmission of economic fluctuations. Without "shocks" the system would lead to all variables assuming equilibrium values depending on the constants in the equations. The shocks are provided by additional terms, especially in equa-

tions (2) (autonomous investment, government expenditure) and (3) (import restrictions, institution of tariffs).

Combination of (2) and (3) provides an indication whether the system is stable:

$$m_i = \rho_i x_i \tag{5}$$

where

$$\rho_i = \mu_i \mathfrak{M}_i .$$

We call ρ the "international reflection ratio." It is the ratio between the increase in a country's imports (as effect) and the increase in its exports (as cause). If $\rho > 1$, the country sends back to the rest of the word larger fluctuations than it received. The weighted average $\Sigma_i \sigma_i \rho_i$ indicates whether the international economic system as a whole is stable ($\Sigma_i \sigma_i = 1$).

Under the simplest assumptions $\mathfrak{M}_i = \dfrac{1}{\gamma_i + \mu_i}$ where γ_i = marginal propensity to consume. If we define:

$$\delta_i = 1 - \gamma_i ,$$

we see that $\rho_i \gtreqless 1$ as $\delta_i \lesseqgtr 0$. The sign of δ_i determines the international stability of country i.

No serious complications follow from the introduction of relative price terms in equations (1) and (3). The result of the operation of relative prices is to shift a certain amount of economic activity from one country to another; but, apart from differences in multipliers between different countries, there is no effect on world trade as a whole.

The parameters in equations (1), (2), and (3) were measured by correlation for 25 countries for the interwar period, making allowance for relative prices in (1) and (3) and for known autonomous factors in (2) and (3). The results were on the whole satisfactory, except for the export equations of crop producing countries and the multiplier equations of Western European countries. For the United States, the multiplier was derived from Tinbergen's system (League of Nations 1939) since it was assumed that fluctuations in U. S. income were primarily autonomous with respect to exports.

The range of the values found for ρ was as follows:

ρ	NUMBER OF COUNTRIES
0.20–0.49	5
0.50–0.74	10
0.75–0.90	4
0.91–1.17	6

The value for δ was found in the neighborhood of 0.15 for Western

Europe, the U. S., and Canada, and near zero for Australia, New Zealand, and the Union of South Africa.

The weighted average of the ρ's was found to be about 0.5. It follows that an initial reduction in world trade (caused, for example, by a depression in the U. S.) will lead to a *further* reduction in world trade of the same magnitude through the operation of the international economic system.

DISCUSSION: Certains auditeurs s'étonnent des faibles valeurs des élasticités obtenues, provenant sans doute d'inerties particulières aux importations, mais aussi d'incertitudes considérables, explicables par les difficultés de l'agrégation. Il faudrait sans doute distinguer diverses catégories de marchandises, catégories assez larges pour éviter des effets excessifs de substitution. Dans l'établissement d'une régression linéaire, le tout est lié à chacune des parties: toutefois si les parties sont assez petites, la dépendance peut être négligée.

The Influence of Devaluation on the Domestic Price- and Wage-Levels
J. A. Hartog, Netherlands Economic Institute.

A NUMBER of countries are supposed to devaluate their currencies with respect to the currency of a given country. The changes of the different wage- and price-levels consequent upon this are analyzed by means of a model presented by J. Tinbergen ("Le problème de rareté du dollar," *Revue d'Economie politique*, Janvier, 1948). This model was generalized so as to enable one to trace the repercussions when more than one country depreciates its currency.

If one supposes that the devaluation takes place in a period of less than full employment, Tinbergen's model can be split up into two parts, one describing the price formation, the other determining the level of the variables, which are expressed in real terms.

We shall introduce the following symbols: p_i indicates the level of retail prices in country i; k_i indicates the exchange rate of country i with respect to country n; π^i_j indicates the import quota of country i with respect to country j, measured as the proportion of the value of imports from country j into country i to the gross national product of country i; π^i_l indicates the amount of labor incorporated into the last unit of product, measured as the quotient of the wage bill and the gross national product; λ_i indicates the absolute increase in the wage level upon a rise in the price-level of one unit. The magnitude of λ is determined by negotiations between trade unions, entrepreneurs, and the government.

If we measure the p_i and k_i in such units that their level just before the devaluation is unity and take this level as origin, with the equations describing the price formation in the form of price-fixing equations, and if we eliminate the wage rates, take the differences and linearize the system by neglecting products of differences, the system may be written as follows:

$$(1 - \lambda\pi_i')\Delta p_i - \sum_j \pi_j'\Delta p_j = - \sum_j \pi_j'\Delta k_i + \sum_j \pi_j'\Delta k_j$$

$(i = 1, 2, \ldots, n)$, $(j = 1, 2, \ldots, n)$, $(j \neq i)$ and $\Delta k_n = 0$. This system expresses the Δp_i in terms of the λ_i and the Δk_i.

The actual computations concerned six countries (the United Kingdom, France, Belgium, the Netherlands, Germany, the United States) and the rest of the world, the United States figuring as country n. For this purpose different hypotheses were made with respect to the magnitude of the λ_i and the different degrees of devaluation Δk_i.

DISCUSSION: Dans quelle mesure le résultat obtenu dépend-il de l'estimation du coefficient lambda lié aux variations des contingences politiques? D'autre part. les inerties sont très différentes selon les branches de production et il en résulte une incertitude sur la signification du résultat.

Quelques applications á l'économie de la théorie mathématique des réseaux, GEORGES TH. GUILBAUD, Institut de Science Economique Appliquée, Paris.

L'ANALYSE théorique et statistique des comptabilités sociales, la multilatéralisation des clearings internationaux, diverses questions concernant la circulation monétaire et l'inflation, conduisent à utiliser des représentations graphiques en forme de réseaux, et par conséquent font souhaiter les développements adéquats d'une théorie générale.

Un réseau est constitué par des cellules ou *comptes* et par des chemins ou *transactions*; chaque transaction *sort* d'un compte et *entre* dans un autre. De ces quatre notions fondamentales et de leurs relations naît une géométrie, qui se double d'une algébre lorsqu'on introduit les valeurs numériques des transactions et qu'on fait quelques hypothèses sur ces valeurs (hypothèses de conservativité, par exemple).

En examinant quelques problèmes particuliers—corrections statistiques d'une comptabilité qui devrait être conservative; calcul indirect de certaines transactions, considérées comme des soldes; formes diverses d'une combinaison de transactions, tel le Revenu National; analyse de l'évolution d'une comptabilité; réduction de créances par clearings multilatéraux; la théorie générale des multiplicateurs, etc.—on est conduit à donner un relief particulier à la notion de *circuit* et on retrouve la dualité noeud-maille découverte par Kirchhoff et qui a joué un très grand rôle en topologie combinatoire.

Mais, dans tous les problèmes précédents, on pose une certaine homogénéité des transactions, qui sont toutes des sommes monétaires. Or, le système économique comprend aussi des flux de biens et services. Il est intéressant de présenter la théorie générale de l'équilibre (celle de Walras et Pareto, mais le cadre est bien plus général) comme une théorie de réseau: cellules de consommation, cellules de production, marchés, et les transactions entre ces éléments.

En commençant par le cas le plus simple, qui est le cas classique, on dotera chaque cellule de production d'une liaison entre les transactions qui y aboutissent (fonction de production), chaque cellule de consommation d'un indicateur de choix, fonction des transactions de la cellule, enfin, on écrira la conservation des flux aux cellules de marché. On peut alors définir d'un seul coup les situations parétiennes optimales en termes réels, on trouve encore des conditions qui font intervertir les mailles. En d'autres termes, l'équilibre général est équivalent à l'équilibre des mailles indépendantes.

Une fois défini l'équilibre en termes réels, on peut montrer comment les flux de biens et services d'une situation optimale peuvent être induits par une circulation monétaire, chaque cellule pouvant alors avoir en une certaine mesure, un comportement autonome de maximum lié.

L'avantage d'une telle présentation est de montrer les extensions possibles de la théorie parétienne; on peut généraliser les fonctions de production en introduisant des liaisons multiples, les variables ne sont plus forcément les transactions d'une même cellule (*external economies*). D'autre part, dans la réalisation d'une certaine autonomie de cellules, le mécanisme connu sous le nom de concurrence parfaite n'est nullement indispensable. On sera amené à définir, pour chaque cellule, un horizon économique qui contient tous les éléments de choix; dans certains cas les horizons seront disjoints, dans d'autres il y aura des recouvrements qui généralisent les situations dites de monopole bilatéral, et établissent le lien entre la théorie des réseaux et la théorie des jeux.

DISCUSSION: Il faut manier avec prudence les analogies, par exemple électriques. Les discussions économiques devraient ne pas perdre de vue l'enchaînement circulaire des causes et des effets; les raisonnements sur les "mailles" sont au moins aussi importants que les raisonnements traditionnels sur les "noeuds." On souligne l'intérêt, pour les économistes, d'instruments mathématiques spécialement adaptés à leur recherche.

Tarification au coût marginal des demandes périodiques, MARCEL BOITEUX, Service Commercial National d'Electricité de France.

LE PROBLEME des "pointes" de la demande est un problème bien connu des cheminots et des électriciens: la demande d'énergie, comme la demande de transport, présente des périodicités marquées (quotidienne, hebdomadaire, saisonnière); au cours de chaque période, la demande passe par un maximum, la "pointe," qui commande le volume des installations et par conséquent celui des coûts fixes.

Le phénomène n'est pas particulier aux entreprises de transport et d'électricité: le problème de la périodicité se pose dès lors que les frais de stockage nécessaires pour adapter le régime de la production à celui

de la demande ne sont pas négligeables, et à fortiori lorsque ce stockage est impossible.

Taille optima des installations. Nous nous bornerons ici au cas d'une production non stockable. Pour simplifier l'exposé, nous admettrons que les facteurs de production se divisent nettement en facteurs mobiles et facteurs fixes que nous représenterons respectivement par un seul facteur, le facteur "exploitation" et le facteur "installation."

Si les régimes de la production et de la demande étaient permanents, la taille optima des installations serait caractérisée par l'égalité du coût marginal d'exploitation et du coût marginal "total" (exploitation + installation). Lorsque le régime de la demande est périodique, on peut décomposer la demande de la période en demandes élémentaires successives sensiblement permanentes; la taille optima des installations est alors caractérisée—en première approximation seulement—par l'égalité du coût marginal total à la moyenne des valeurs prises par les coûts marginaux d'exploitation relatifs au volume de chaque demande élémentaire.

Cas des installations inélastiques. Les installations habituelles sont "inélastiques": la dépense de production croît proportionnellement à la quantité produite q jusqu'à une quantité limite q_1 qu'il est impossible de dépasser. Pour $q = q_1$, le coût marginal d'exploitation n'est ni égal au coût partiel, ni infini, mais *indéterminé*; la courbe du coût marginal d'exploitation en fonction du débit affecte donc la forme d'un angle droit dont une branche est pratiquement horizontale (d'ordonnée égale au coût partiel), et dont l'autre est verticale (d'abscisse égale à la capacité limite de l'installation).

Lorsque l'installation a la taille optima, l'une au moins des demandes élémentaires de la période coupe la courbe du coût marginal d'exploitation sur sa branche verticale. Tarifiées au coût marginal, les demandes qui coupent la branche horizontale ne rémunèrent que les dépenses partielles; les autres rémunèrent en outre les frais d'installation (évaluées au coût marginal).

Conclusion et généralisation. Si, conformément à la terminologie des électriciens, on appelle "courbe de charge" la courbe donnant, au cours de la période, la quantité demandée en fonction du prix, la tarification marginaliste des pointes d'une installation inélastique a pour objet d'écraser cette courbe suivant une horizontale partout où ce résultat peut être obtenu sans faire descendre le tarif horaire au-dessous du coût partiel. La courbe de charge, après tarification, est donc formée d'une succession de paliers de même niveau liés entre eux par des portions de courbe situées en dessous de l'horizontale des paliers. L'ordonnée de l'horizontale est égale à la capacité limite de l'installation. Cette capacité est optima lorsque le produit des tarifs qui assurent l'écrasement de la

courbe de charge, rémunère, outre les dépenses partielles, le coût marginal des installations.

L'horizontale des paliers est le diagramme de "productibilité" de l'installation. Lorsque la productibilité subit des variations périodiques (centrales hydro-électriques au fil de l'eau), le résultat ci-dessus se généralise: la courbe de charge après tarification doit épouser la forme du diagramme de productibilité sauf en certaines régions où elle présentera des décrochements.

DISCUSSION: L'expérience T.V.A. est invoquée ainsi que les discussions de tarifs qui ont eu lieu à cette occasion. Les effets de substitution modifient le mécanisme de l'étalement des pointes mais ne changent pas l'aspect formel du résultat. Les demandes horaires sont-elles élastiques? Si elles ne le sont pas toutes, ce sont justement les demandes inélastiques qui supporteront les frais d'installation.

Decision Models, RAGNAR FRISCH, University Institute of Economics, Oslo.

L'AUTEUR étudie un modèle où se trouvent représentés les principaux moyens d'action du gouvernement; en réduisant le système à une soixantaine de variables, il reste douze degrés de liberté qui décrivent les possibilités de décisions politiques,. Il importe de souligner les conditions-limites et les discontinuités. Le choix peut être figuré par un point mobile dans un domaine polyédrique dont les frontières et la connection doivent être étudiées de très près.

Vilfredo Pareto vu par le profil Keynésien, ERALDO FOSSATI, Università, Trieste.

LE TEMPS est désormais venu de pouvoir affirmer qu'il est sans signification de continuer à discuter en faveur ou contre l'oeuvre de Keynes, une des figures auxquelles, il faut bien le reconnaître, nous devons le plus de progrès dans la science économique; le nier ou l'opposer à ses prédécesseurs, c'est nier le processus naturel du développement de notre science. Pour nous convaincre d'une vérité aussi élémentaire, il convient, à notre sens, de nous poser la question suivante: quelle vision pourrions-nous avoir de l'oeuvre de Pareto si nous la voulions voir à travers des lunettes Keynésiennes?

Pareto représente l'expression la plus haute de la théorie statique et Keynes l'expression la plus vive du travail moderne de création de la théorie dynamique. Entre l'équilibre du premier, impliquant le plein emploi de tous les facteurs de production et l'équilibre sans plein emploi du second, on ne saurait envisager de comparaison qui ferait ressortir le contraste entre deux terrains de recherche aussi différents que statique et dynamique. Il reste seulement à voir le point de jonction des deux systèmes, ou mieux encore, la signification de l'un par rapport à l'autre dans le processus du passage de la statique à la dynamique.

D'un tel point de vue, nous pourrions, je pense, arriver aussi à nous représenter la "théorie générale" de Keynes comme un cas "particulier" de la "théorie dynamique générale," que l'on peut dériver de l'oeuvre parétienne. Nous aimerions voir le passage de la statique à la dynamique par le recours aux deux principes parétiens fondamentaux suivants qui, semble-t-il, confèrent bien à Pareto le droit d'occuper une place éminente, même en dynamique: (a) aucune place ne peut être assignée à la théorie de la monnaie dans la théorie de l'équilibre général; (b) la logique constructive de la statique et de la dynamique économique doit être une.

Partant de ces deux principes généraux, nous avons pu retrouver, il y a aujourd'hui plus de quinze années, grâce à l'introduction systématique des facteurs "incertitude" et "risque" (interprétés d'une manière quantitative tout à fait particulière) dans la théorie de l'équilibre économique général, une théorie dynamique *générale* dont la théorie statique serait le cas-limite. Par cette voie, nous pouvons aussi nous représenter un cas "particulier" d'équilibre dynamique sans plein emploi par simple déduction de l'hypothèse d'un cas spécial dynamique dans lequel les fonctions d'excès de demande soient fonctions homogènes de degré zéro relativement aux prix. On pourrait prétendre que ce ne serait pas là recourir exactement au processus keynésien, mais il semble que l'on peut bien affirmer que notre résultat est le résultat keynésien naturel dans le tableau de la théorie dynamique dérivée de la théorie de l'équilibre général. Ainsi pourrait être également établie la fonction évolutive et non révolutionnaire de l'oeuvre keynésienne dans le développement de la science économique.

Quelques observations sur la complexité et la simplicité dans la recherche économétrique, Hans Staehle, Economic Commission for Europe.

Alors qu'au début, l'économétrie moderne s'occupait en premier lieu de problèmes micro-économiques (notamment la mensuration par Frisch et Irving Fisher de l'utilité marginale de la monnaie), son développement ultérieur, fortement stimulé par les idées de Keynes, a porté de plus en plus sur les recherches macro-économiques. En même temps, l'usage des mathématiques s'est intensifié, et l'on a profité de méthodes statistiques nouvelles qui, elles aussi, ont une base mathématique beaucoup plus étendue que celles connues il y a vingt ans.

Ce développement est-il nécessaire? En d'autres mots, peut-on considérer comme économétriques au sens propre des études qui portent sur une partie seulement d'une économie nationale, voire de l'économie mondiale? Des résultats quantitatifs obtenus dans un champ restreint par des méthodes simples, sans recours aux mathématiques ou aux techniques statistiques avancées, sont-ils admissibles?

La réponse à ces questions est évidente. On peut la formuler en citant Pareto ou Barone: le seul but de la théorie est d'expliquer les faits,

pourvu qu'on entende par "faits" l'aspect objectif ou quantitatif des choses. Il importe peu que ces faits soient micro- ou macro-économiques. On peut ajouter que la "généralité" et l'"élégance," vertus essentiellement mathématiques, ne sont pas des vertus en soi lorsqu'il s'agit de comprendre ou d'expliquer un ensemble de faits économiques. Au contraire, on risque de perdre davantage en réalisme que ce qu'on gagne en généralité. Quant aux méthodes de recherche à appliquer, une seule restriction paraît nécessaire: qu'elles soient aussi simples que possible pour être efficaces.

Pour soutenir ces thèses, on présentera des applications d'un théorème de Marshall, celui de l'inélasticité de la demande indirecte (inelasticity of derived demand), à des "industries," applications conduites au moyen de procédés essentiellement graphiques.

Ces applications, dues aux efforts de deux jeunes étudiants de l'Université de Harvard, Roger Smith et Harold Pilvin, concernent, d'une part, une industrie particulière (à savoir celle des fournitures pour plombiers) et, d'autre part, la conduite d'un certain nombre d'industries, en ce qui concerne leurs prix de vente, pendant la dépression 1929–1933, en fonction du "degré de monopole."

Les détails de ces recherches et leurs résultats seront présentés oralement, avec documents à l'appui. Il en résultera que ces résultats méritent d'être acceptés avec confiance comme explications satisfaisantes de certains faits économiques.

Sans vouloir le moindrement diminuer l'intérêt que présentent les développements techniques sus-mentionnés, on pourrait être tenté de penser que, même dans l'étude de situations relativement complexes, des moyens simples sont parfois suffisants pour dégager des liaisons intéressantes. En général, on peut se demander s'il n'y aurait pas lieu de mettre en garde contre une certaine tendance vers l'emploi de théories et de méthodes à caractère technique souvent très poussé, dont l'application, à moins qu'elle ne soit nécessaire, pèche lourdement par manque d'élégance. D'autre part, les recherches portant sur un champ restreint où les faits peuvent encore être compris comme résultant d'actions individuelles, ont certainement l'utilité de rappeler à l'esprit la complexité extraordinaire des variables de la macro-économie où les décisons individuelles des consommateurs et des producteurs sont noyées dans l'anonymat des phénomènes collectifs.

DISCUSSION: Convient-il de mettre en oeuvre tous les raffinements de l'outil statistique, alors que le matériel de base reste grossier? Quelle foi peut-on accorder à un résultat qui n' apparâit que grâce à l'application de méthodes complexes? La discussion porte ensuite sur l'influence du degré de concentration sur les prix (thèse de Mr. Harold Pilvin) et l'évolution comparée de la plomberie et du bâtiment (Roger Smith).

Le coût marginal, ROGER HUTTER, Société Nationale des Chemins de Fer Français.

LE COÛT marginal est défini comme la variation des dépenses de production avec la production elle-même. Si les dépenses étaient une fonction mathématique continue et dérivable de la production, le coût marginal en serait la dérivée; mais la réalité est beaucoup plus complexe. Il faut tout d'abord rapprocher les dépenses de la production correspondante, ce qui se fait pratiquement par *exercices*. Il faut tenir compte du fait que les dépenses sont engagées en fonction de la production *prévue* et que si la production réelle est inférieure à cette prévision, certaines dépenses subsistent à leur montant primitif: elles sont *inéludables*. Il faut tenir compte du décalage temporel entre dépense et production par un processus spécial, appelé *amortissement*. Enfin, de nombreux facteurs de production ne peuvent être mis en oeuvre que par quantités indivisibles: si ces quantités sont petites, on peut valablement les *moyenner*, mais si elles sont grandes, la dépense présente une discontinuité.

Compte tenu de ces constatations comptables banales, l'évolution des dépenses d'une entreprise produisant un seul bien s'étudie aisément, et l'on constate que les dépenses de production d'un exercice dépendent à la fois de la production réelle et de la production prévue. Si l'on examine plusieurs exercices successifs, on est amené à constater que le point représentatif des dépenses en fonction de la production se déplace sur un faisceau de lignes limité par une enveloppe inférieure correspondant au plein emploi de tous les facteurs de production. On est amené, de ce fait, à considérer trois coefficients: (a) *le prix de revient partiel* ne tenant compte que des dépenses immédiatement éludables et qui vaut pour les variations très rapides, hors du plein emploi; (b) *le coût marginal partiel* qui tient compte en plus de dépenses lentement éludables et qui vaut pour les variations lentes, hors du plein emploi; (c) *le coût marginal total* qui tient compte de toutes les dépenses et qui ne vaut que pour les variations lentement croissantes, avec permanence du plein emploi.

L'étude du long run est inséparable de celle du *progrès technique*; celui-ci se présente le plus généralement sous la triple forme simultanée de: la diminution du prix de revient partiel; l'augmentation des dépenses fixes; l'augmentation de la production maximum. Le progrès technique est donc, le plus souvent, lié à un niveau minimum de production, au dessous duquel il se renverse. En outre, si la production est constante, le progrès technique provoque un certain sous-emploi, car il a pour sous-produit gratuit un excédent de capacité de production, inutile dans ce cas.

Dans la plupart des entreprises peuvent se présenter toute une série de circonstances qui compliquent le calcul du coût marginal: Certains facteurs de production peuvent présenter de grandes indivisibilités, en-

traînant des discontinuités dans la variation des dépenses; l'entreprise peut être astreinte à satisfaire la demande, ce qui l'oblige à avoir des marges de sécurité; la demande peut avoir des oscillations périodiques, ce qui pose des problèmes très délicats; le bien produit peut être stockable; l'entreprise peut produire simultanément plusieurs biens; certains de ces biens peuvent faire l'objet de productions parfaitement ou imparfaitement liées. Enfin, l'économie dans laquelle vit l'entreprise peut être distordue, c'est-à-dire que diverses causes éloignent le rendement social de son maximum, et que les prix—et notammment ceux des facteurs de production—sont faux.

Ces indications permettent de saisir les difficultés extrêmes que rencontre un chef d'entreprise désireux de connaître les coûts marginaux de ses productions; la recherche intelligente du coût marginal nécessite une connaissance exceptionnellement bonne de la structure intime de l'enterprise, et de ses évolutions virtuelles, ainsi qu'une culture mathématique très étendue. Aussi, le coût marginal, dans son acception originelle, est-il un concept abstrait; on ne peut le chiffrer concrètement qu'en simplifiant considérablement son calcul par une série de conventions.

DISCUSSION: Brève discussion sur la possibilité de voir la courbe de dépense descendre jusqu'à la courbe de long terme en cas de lente régression de l'industrie. On souligne l'intérêt de ces confrontations des notions théoriques avec les préoccupations des techniciens.

Remarques sur une notion d'utilité générale et une définition de prix rationnels, Noël Bachet, Ingénieur en Chef des Transports.

Un individu qui dispose d'un revenu r et achète des biens directs aux prix p, choisit la répartition qui lui donne l'utilité maxima. Il rend donc maxima sa fonction d'utilité $\varphi(q_1, q_2, \cdots)$ avec la condition: $\Sigma\, pq = r$. Les propriétés mathématiques de la solution sont bien connues. On en déduit qu'à partir d'un complexe adapté aux prix p—c'est-à-dire résultant d'un choix libre en présence de ces prix—la *variation d'utilité individuelle* $d\varphi$ est proportionnelle à la différentielle $\Sigma p\, dq$ quelles que soient d'ailleurs les variations dq, libres ou imposées. Cette variation est homogène à un revenu. Puisque l'on a toujours: $\Sigma p\, dq + \Sigma q\, dp = dr$, on voit qu'une variation d'utilité $\Sigma p\, dq$ donnée est équivalente à celle que donnerait un accroissement de revenu $dr = \Sigma p\, dq$, les prix étant laissés constants. On voit apparaître une mesure objective de la variation d'utilité. Il ne faut pas toutefois se méprendre sur cette notion. Dire que les variations $\Sigma p\, dq$ de deux individus sont égales, cela signifie seulement que, pour l'un et pour l'autre, elles sont équivalentes à une même variation de revenu $dr = \Sigma p\, dq$. L'égalité n'est que dans la cause susceptible de les produire, non dans les effets subjectifs. Il faut utiliser la notion ainsi découverte d'une manière pragmatique. Cette notion peut donner lieu néanmoins à des applications intéressantes.

Si on considère deux états économiques très voisins A et B, tels que le passage de A à B apporte une variation $\Sigma p\,dq$ nulle ou positive pour tout individu, on a: $\Sigma(\Sigma p\,dq) = \Sigma p\,dQ \geqslant 0$, Q étant les quantités totales des biens directs fournies aux foyers. La réciproque n'est pas vraie. On peut avoir $\Sigma p\,dQ > 0$ sans que toutes les variations $\Sigma p\,dq$ soient positives. Mais si, grâce à une amélioration de la production un état B apporte par rapport à un état A une variation $\Sigma p\,dQ$ positive, où certaines variations individuelles $\Sigma p\,dq$ sont négatives, on peut modifier la répartition des revenus de telle manière que chaque variation $\Sigma p\,dq$ soit positive, la variation totale $\Sigma p\,dQ$ étant désormais participée par tous. Par ailleurs, on remarquera que si l'on est dans un état de maximation du rendement social, il est impossible, en vertu même de la définition de cette notion, de modifier l'état économique en ayant des variation $\Sigma p\,dq$ toutes positives ou nulles. Il en résulte que, dans cet état, en respectant les conditions données, toutes les variations possibles sont telles que l'on a: $\Sigma p\,dQ = 0$.

Puisque l'on est par hypothèse dans un régime de libre choix, à défaut de quoi la différentielle $\Sigma p\,dq$ n'aurait aucun sens, toute modification de structure se manifeste par des variations de prix. Les prix qui suscitent un emploi optimum d'un potentiel de production donné sont tels que toute modification de ces prix avec maintien du plein emploi, laisse nulle la variation $\Sigma p\,dQ$. Ces prix sont donnés par la formule:

$$p = a\frac{dA}{dQ} \qquad \frac{dB}{dQ} + \cdots,$$

a, b, étant des prix de facturation qui ajustent l'offre et la demande et A, B, les consommations des facteurs de production. Le sens des quotients dA/dQ, dB/dQ, appelle des remarques intéressantes.

Cette notion de prix rationnels peut s'étendre aux prix des biens indirects. Dans le cas d'application de ces prix, on peut donner une portée plus générale au sens de la différentielle. Ces notions permettent d'aborder de nombreux problèmes, notamment en ce qui concerne la tarification rationnelle.

DISCUSSION: L'exposé est rattaché aux discussions classiques sur la théorie des surplus et la constance de l'utilité marginale de la monnaie On souligne que toute décision concernant le bien-être collectif implique le dépassement du "no bridge."

La notion de surface de coût dans l'étude du comportement dynamique des entreprises, MICHAEL J. VERHULST, International Bank for Reconstruction and Development.

EN TOUTE rigueur, la théorie des fonctions de production ne s'applique qu'à une entreprise considérée individuellement et n'explique que le comportement technique de l'entreprise.

Par ailleurs, une entreprise est très souvent composée de multiples

organes caractérisés chacun par une fonction de production dite élémentaire; ainsi, une batterie de fours à coke, un haut-fourneau dans un complexe métallurgique. Ces fonctions de production élémentaires sont le plus souvent linéaires. On peut alors montrer qu'en première approximation la fonction de production de l'entreprise est également linéaire.

On peut aussi montrer que pour une entreprise de structure donnée, la quantité produite, représentée par la fonction de production, peut se ramener à une fonction des degrès d'emploi des capacités des différents organes qui composent l'entreprise. On peut montrer de plus que cette quantité étant représentée, en fonction des quantités utilisées des divers facteurs, par la surface de production, il y a sur cette surface, pour chaque quantité produite, un point d'équilibre où le coût est minimum pour une production donnée, et que lorsque cette quantité varie, le lieu du point d'équilibre est une "courbe de coût minimum" tracée sur cette surface.

Si l'on considère ensuite comme variables, non seulement les degrès d'emploi des divers organes de l'entreprise, mais encore leurs capacités, ce qui est le cas dans un milieu en expansion qui exige un développement de l'entreprise, on peut regarder la quantité produite comme une fonction de ces capacités en même temps que de leur degré d'emploi. Nous appelons cette fonction "fonction de production dynamique." On peut observer que la fonction de production dynamique d'une entreprise se réduit à la fonction ordinaire de production quand on considère les capacités comme des constantes.

On peut donner à cette fonction de production dynamique une forme synthétique générale. Désignant par u le degré d'emploi et par h la capacité (ou encore le capital de l'entreprise, puisque l'un est un bon indice de l'autre), la fonction de production est du type:

$$y = F(u, h), \tag{1}$$

où y est la production de l'entreprise exprimée en volume physique.

Le coût de fabrication technique C_y de la production y est, de la même façon, fonction du degré d'emploi de la capacité de l'entreprise et du capital de l'entreprise, de sorte que:

$$C_y = G(u, h). \tag{2}$$

Eliminant u entre les deux fonctions, on constate que le coût de fabrication technique C_y est fonction de la production y et du capital h:

$$C_y = H(y, h), \tag{3}$$

où y est une variable de débit et h une variable de structure. On est amené ainsi à représenter le coût de fabrication technique par une "surface de coût technique" exprimant le coût en fonction d'une *variable de structure* et d'une *variable d'activité*.

On peut admettre qu'en première approximation le coût de la main-d'oeuvre, le coût d'entretien et les frais généraux sont proportionnels au coût de fabrication technique C_y, de sorte que si C désigne le coût total, on peut aussi écrire:

$$C = C(y, h). \tag{4}$$

La surface représentée par (4) peut être appelée "surface de coût."

Les points de la surface de coût correspondant à une valeur donnée du degré d'emploi vérifient à la fois l'équation (4) [ou (3)] et l'équation (1) [ou (2)], c'est-à-dire qu'ils se disposent sur une courbe dite "courbe d'échelle" exprimant le coût en fonction du débit ou de la capacité. Alors que la courbe ordinaire de coût exprime celui-ci en fonction du débit (ou du degré d'emploi) pour une capacité donnée, il existe une courbe d'échelle pour chaque valeur donnée du degré d'emploi.

Nous avons pu déterminer numériquement la surface de coût pour l'industrie des transports aériens et l'industrie du gaz en France, et pour l'industrie du pétrole aux Etats-Unis.

DISCUSSION: Les résultats sont confrontés avec ceux obtenus par le Professeur Schneider dans son étude sur l'industrie du chocolat. La discussion porte ensuite sur la nature de la courbe de long terme étudiée par l'auteur: contour apparent de la surface de coût ou bien courbe historique?

La théorie marginaliste et l'organisation scientifique du travail, JEAN ROCHER, Ingénieur des Arts et Manufactures.

INTRODUCTION. Rappel du théorème de Gossen: "Lorsque ses ressources sont insuffisantes pour lui permettre de se procurer tout le bien possible à satiété, l'homme doit se procurer chacun d'eux dans les proportions telles que la valeur d'usage du dernier atome soit la même pour tous les biens." Démonstration du théorème par la méthode de Monsieur Tintner: dans le cas du régime permanent; dans le cas du régime non permanent, avec définition du terme correcteur dynamique.

Application de la théorie marginaliste à l'organisation scientifique du travail: définition du rôle de l'organisation scientifique du travail; possibilité d'appliquer le théorème de Gossen à ce problème, du fait du caractère toujours limité du budget d'une entreprise, et un caractère mesurable de l'utilité marginale dans le cas du problème d'organisation; remarque sur le rôle des contingentements; interprétation des différents termes et plus particulièrement du coefficient correcteur dynamique; importance de la notion d'horizon.

Conséquences du théorème de Gossen dans l'organisation scientifique d'une entreprise: limite d'emploi d'un agent de la production; rétribution d'un agent de la production; coût réel des services d'un agent de la production; prix de revient réel du fonctionnement d'un organisme; coefficient

de productivité—son importance pour l'évaluation du degré d'organisation de l'entreprise et de la prospérité de cette entreprise; nouvelle définition du rôle de l'organisation et les conditions à remplir pour que le but recherché soit atteint.

Quelques conséquences sociales de la théorie marginaliste appliquée à l'organisation scientifique du travail. A l'intérieur de l'entreprise: position réelle de l'entreprise vis-à-vis de la limitation de son budget; caractéristiques des rôles des différentes catégories d'agents (*techniciens:* diminuer l'utilité marginale des agents de la production; *chercheurs:* augmenter cette utilité marginale; *commerçants:* augmenter l'utilité marginale des produits; *financiers:* augmenter les disponibilités budgétaires); la hiérarchie (des valeurs, des horizons, des utilités marginales, des salaires).

Conséquences sociales plus générales: opposition entre les exigences des lois de répartition optima des richesses et les exigences de la production de ces richesses; causes d'opposition de l'intérêt général et des intérêts particuliers du fait: de la différence d'horizon; de l'intérêt de l'individu de conserver une utilité marginale élevée, et de la Société de diminuer cette utilité marginale.

Conclusion. Difficultés d'évaluation des utilités marginales du fait: de la grande élasticité de l'utilité marginale; de l'interdépendance des utilités marginales des différents agents de la production; du caractère dynamique des problèmes posés; possibilités cependant de solutions au moins approchées dans le cas des problèmes d'organisation; difficultés d'une extrapolation à des ensembles plus importants que les entreprises et précautions à prendre.

DISCUSSION: Le terme dynamique correctif est rapproché des équations eulériennes du Professeur Amoroso. Quelques précautions doivent être prises dans l'usage et l'interprétation de cette vitesse de variation du coût marginal. Des modèles simples et discontinus permettent peut-être de donner une signification intuitive au terme correctif.

Répartition rationnelle des stocks de marchandises dans un réseau de distribution, Robert Hénon, Institut de Statistique de Paris.

La statistique mathématique permet de décrire d'une manière précise les lois de l'*écoulement aléatoire* de divers stocks de marchandises. Ces lois conduisent à définir par nature de stock un *niveau de "sécurité"* à partir duquel il faut se réapprovisionner pour satisfaire la demande avec une probabilité fixée à l'avance. L'application de cette méthode de gestion entraîne l'existence d'un stock mort qui est mesuré par l'espérance mathématique du stock restant à la fin du délai de réapprovisionnement. Le calcul du stock mort global d'un *réseau de distribution* permet de choisir la chaîne de distribution optima comportant ou non la présence d'un ou plusieurs grossistes interposés entre le producteur et les détaillants.

Ecoulement d'un stock. La distribution de fréquence des sorties peut être étudiée comme dans le cas d'écoulement des eaux de rivières (Gibrat) en prenant comme unité de durée élémentaire le jour ou la semaine. Les lois dissymétriques semblent généralement apparentées à la loi de Galton MacAlister. Si l'on fait intervenir le délai de réapprovisionnement Δ, multiple de la durée élémentaire, la nouvelle loi de distribution pour ce délai Δ est une somme de variables aléatoires que l'on trouve à partir de la fonction caractéristique. Graphiquement si l'on prend Δ comme variable, une loi Gaussienne en première approximation, le lieu des points du stock final attendu dans 95% des cas est une surface interceptée par une courbe parabolique; on en déduit le *stock de sécurité* et le *stock mort* de la forme $2s\sqrt{\Delta}$ (au second ordre près).

Comparaison de deux réseaux de distribution. Considérons une "région" sans grossiste comprenant N détaillants ayant mêmes débits de marchandises et mêmes délais d'approvisionnement les stocks morts sont mesurés par $ts\sqrt{\Delta}$. Le "stock mort" global de cette branche de réseau est: $tNs\sqrt{\Delta}$. Si un centre livreur est installé au milieu de la région pour desservir les N détaillants, l'écart-type du débit aléatoire de marchandises devient $s\sqrt{N}$ et son stock mort: $ts\sqrt{N\Delta}$. Par contre, les détaillants étant approvisionnés par ce centre en un temps très court égal à l'unité (par exemple $\Delta = 1$ jour), le stock mort global devient:

$$tsN + ts\sqrt{N\Delta}.$$

L'efficacité de l'intervention d'un centre livreur ou grossiste peut être mesurée par le rapport du nouveau stock mort à l'ancien:

$$\frac{tsN + ts\sqrt{N\Delta}}{tNs\sqrt{\Delta}} = \frac{1}{\sqrt{\Delta}} + \frac{1}{\sqrt{N}}.$$

Par exemple, si $\Delta = 36$ jours, $N = 100$ détaillants, le nouveau stock mort est de $\frac{1}{6} + \frac{1}{10} = 0.267$ de l'ancien.

L'intervention du prix de transport permet encore de trouver le prix de revient final de la marchandise et de calculer la dimension optima des *lots de renouvellement* adaptés aux réseaux.

DISCUSSION: Ces considérations théoriques sont de nature à éclairer le dilemme entre distribution directe et système de centres livreurs, où la pratique manifeste souvent quelques hésitations. Le problème des stocks de marchandises évoque celui des encaisses monétaires.

The Cobb-Douglas Production Function: Interpretation of the Statistical Results. D. H. van Dongen Torman, Netherlands Economic Institute.

Starting from the theory of marginal productivity, written in the form: $\Delta y_0/\Delta y_1 = 1$, where $y_0 = p_0 x_0$, p_0 standing for price of product and x_0 for

quantity of product, and where $y_i = p_i x_i$, p_i standing for prices of factors of production and x_i for quantities of factors of production, it can be shown that $p_1 \neq p_0(\partial x_0/\partial x_i)$. Even when assumptions are introduced regarding free competition in the markets for product, labor, capital, capital goods, and raw materials, when the latter are taken to be limitational with respect to the product, the prices of the factors of production cannot be explained by this last expression when the variables are measured in physical units. On the whole no such relation will be found when measuring in value units.

As regards the distribution of income, assumed to be in the proportion of the exponents in the production function, it can be shown that again a great number of assumptions have to be made when the variables are measured in physical units. When measured in value units, however, no such assumptions as mentioned above are necessary.

A good fit between real and computed income distribution in the latter case, therefore, does not allow conclusions such as are often drawn by Douglas, et al.

Les indices de rationalisation, René Roy, Institut de Statistique de l'Université de Paris.

D'une manière générale, nous désignons par "indices de rationalisation" un nombre indice qui exprime numériquement le degré de perfectionnement atteint dans une branche particulière d'activité ou dans un groupe éventuellement plus étendu, par rapport à une situation de référence.

Si nous considérons par exemple le cas simplifié d'une opération qui n'exige que l'utilisation de main d'oeuvre à l'exclusion de tout autre facteur de production, et qui s'applique à la fabrication d'un seul genre d'articles, nous pouvons définir l'indice de rationalisation relatif à cette opération particulière au moyen du rapport q/h de la quantité produite au nombre d'heures de travail consacrées à cette production. Si nous admettons en outre que la branche d'activité en cause est régie par la concurrence parfaite, le rapport qui précède pourra être remplacé par le quotient s/p du salaire au coût de production.

Plus généralement, lorsque nous envisageons une opération qui exige, outre la main d'oeuvre, l'utilisation d'autres facteurs de production et qui aboutit par ailleurs à la production simultanée de plusieurs genres différents d'articles, nous devons remplacer l'expression simplifiée qui précède par le rapport de deux nombres indices, l'un concernant le volume de la production et l'autre, celui des facteurs mis en oeuvre. Dans le cas de concurrence parfaite, mais dans ce cas seulement, l'indice de rationalisation peut être également défini par le rapport de l'indice du coût des facteurs à l'indice du coût de production pour la branche d'activité en cause.

Le calcul pratique des indices de rationalisation ainsi conçus pose donc les mêmes problèmes que celui des indices de prix et de quantités; ils se résument essentiellement: en premier lieu, à la définition du champ des opérations embrassées par l'indice à calculer, et en second lieu, par le mode de calcul auquel on entend recourir.

Sur ces deux points, nous proposons de faire appel aux solutions que nous avons suggérées dans un mémoire antérieur consacré aux "index économiques" et reposant sur l'utilisation généralisée des indices de type monétaire, c'est-à-dire des indices à chaine pouvant être assimilés en fait à des indices budgétaires comportant l'utilisation de poids variables et par conséquent la référence à des situations intermédiaires entre la situation courante et la situation de référence.

Une telle méthode présente l'avantage de rattacher les indices de rationalisation à un système cohérent d'indices de prix et de quantités concernant soit des opérations élémentaires, soit des ensembles plus ou moins étendus. Parmi les questions auxquelles s'apparente la recherche des indices de rationalisation figurent notamment celles qui concernent le salaire réel, les élasticités de production pour la main-d'oeuvre ou pour le capital et les éléments constitutifs du revenu national.

DISCUSSION: L'introduction d'un taux de rationalisation caractérisant le rythme du progrès technique au cours du temps met en évidence l'influence respective des variations de l'effectif des travailleurs, de l'importance des investissements et du progrés technique, dans une formule généralisée de Douglas.

CONSULTATION ACTIVITIES WITH THE UNITED NATIONS

THE ECONOMETRIC SOCIETY was represented at the recent United Nations Scientific Conference on the Conservation and Utilization of Resources by virtue of the consultative status which the Society has with the Economic and Social Council of the United Nations. The report which follows was prepared by Mr. Howard R. Tolley, Director of the Economics and Statistics Division of the Food and Agriculture Organization, who served as the appointed representative of the Society at the conference.

This Conference, which was convened at United Nations Headquarters on 17 August and adjourned on 6 September 1949, was very different from any other conference which has been convened by the United Nations or any of its agencies. The Conference was attended by delegates from some thirty nations, members of the United Nations Organizations, and by representatives of a selected group of learned societies and institutions. They included about 500 "scientists, engineers, resource technicians, economists, and other experts." Conferees participated "as individual experts and not as representatives of their governments or organizations."

There were no political speeches, and no resolutions calling for action by member nations or other international organizations were proposed. Instead, there was an honest effort on the part of all participants to consider in a "scientific" manner the problems involved in the conservation and utilization of the world's resources.

In the three weeks during which the Conference was in session more than three hundred papers were presented and discussed. They covered a very wide range of subject matter. Some examples are: the pressure of population on resources; individual farm planning for optimum use of resources; statistical tools in resource appraisal and utilization; organization of forest services; undiscovered oil and gas reserves; techniques for increasing coal production; tools and equipment for small-scale farming; management and cultivation of fresh water fish.

The Conference was organized in six sections: Minerals, Fuels and Energy, Water, Land, Forests, Wild Life and Fish. In addition to the section meetings, there was a series of so-called Plenary Meetings where papers on topics of interest to the entire Conference were presented and discussed. The proceedings of the Conference are being published by the United Nations. In all, this Conference had many of the characteristics of a meeting of a "learned society" and in spite of the heterogeneity of the topics considered the delegates and participants impressed one as being a rather homogeneous international group bent on exploring the frontiers of their various specialized fields.

To an economist, it was notable, and in a sense disappointing, that even though the Department of Economic Affairs of the United Nations had general responsibility for the organization of the Conference, there was no section devoted to the Economics of Conservation and Utilization, and in comparatively few of the papers was there careful analysis and consideration of economic problems and implications. In fact most of the speakers seemed to consider that "resources" meant natural resources only, e.g., minerals, land, water, etc. Only a few seemed to be cognizant of the modern economic concept of "allocation-of-resources" or even the older concepts of the "combination-of-factors" and "marginal utility." On the other hand, among the few papers presented by economists and statisticians were some that gave the results of pioneering work and at the same time pointed

to the need for much more economic and statistical study and analysis. Notable in this respect were papers by Colin Clark of Australia, P. C. Mahalanobis of India, Frank Yates of England, E. De Vries of Holland, and John D. Black, Sherman Johnson, and V. Webster Johnson of the United States.

With the increasing international emphasis that is being given to "economic development," it is to be hoped that the economists and statisticians of the world will devote more of time and energy than they have in the past to study and analysis on a national and international basis of the allocation of resources, the combination of the factors of production, the economic costs and benefits of resource exploitation, and related subjects that are so important in developing national and international plans and programs. This is both a challenge and a responsibility.

REPORT OF THE COUNCIL FOR 1949

THE COUNCIL approved the election of seven new Fellows in 1949. The names suggested by the Fellows included all the candidates nominated by the Council in 1948 but not elected, and from these names (none having been suggested by other members of the Society) the Council nominated sixteen candidates, three having been tied for fourteenth place. From this list the Fellows elected seven new Fellows as follows: Maurice Allais, Abram Bergson, Bernard Chait, Milton Friedman, M. G. Kendall, P. C. Mahalanobis, and Franco Modigliani. Brief bibliographies of the new Fellows are printed elsewhere in this issue.

The Council elected the following officers for 1950: President, Tjalling C. Koopmans; Vice-President, R. G. D. Allen; Secretary, William B. Simpson; Treasurer, Alfred Cowles.

By unanimous vote of the Council, Dickson H. Leavens, retired managing editor of ECONOMETRICA, was elected to honorary membership in the Society. A brief biography of Mr. Leavens appears in the October 1948 issue of ECONOMETRICA.

The terms of the following members of the Council expired on December 31, 1949: R. G. D. Allen, Colin Clark, Simon Kuznets, and Jacob Marschak. Of these all but Jacob Marschak were eligible for re-election, the latter already having served two consecutive three-year terms. The Fellows elected the following for three-year terms ending December 31, 1952: R. G. D. Allen, Colin Clark, Simon Kuznets, and Herman O. A. Wold. Members of the Council whose terms expire on December 31, 1951 are Costantino Bresciani-Turroni, François Divisia, Wassily Leontief, and Paul A. Samuelson. Members of the Council whose terms expire on December 31, 1950 are Ragnar Frisch, John R. Hicks, and Oscar Lange. The officers of the Society are ex-officio members of the Council.

Changes in dues and subscription rates voted by the Council in 1949 are reported elsewhere in this issue.

WILLIAM B. SIMPSON, *Secretary*

TREASURER'S REPORT

Statement of Accounts, Fiscal Year Ended September 30, 1949

Income

Dues: United States and Canada	$4,202.00		
Other countries	1,660.40	$ 5,862.40	
Subscriptions: United States and Canada	$1,873.00		
Other countries	2,847.60	4,720.60	
Sale of back numbers		2,537.98	
Sale of exchanges		309.99	
Miscellaneous income		101.84	
Total earned income			$13,532.81

Expense

Journal: Salaries	$2,351.04		
Printing and mailing Vol. 17 and provision of inventory copies	4,981.03		
Washington Proceedings Supplement	2,900.00		
Transfer of inventory to new printer, insurance on back numbers, etc.	567.09	$10,799.16	
Administration: Salaries	$3,514.04		
Postage	651.45		
Printing and stationery	502.74		
Miscellaneous	448.04	5,116.27	
General: Audit	$ 150.00		
Sundry	36.68	186.68	
Total expense			16,102.11
Excess of expense			$ 2,569.30
Less net increase in inventory			52.96
Net loss carried to surplus			$ 2,516.34

Balance Sheet as of September 30, 1949

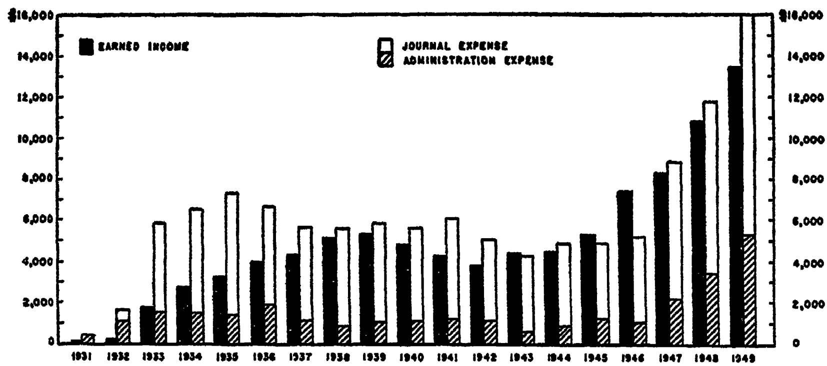
Cash in bank			$ 3,362.8
Accounts receivable:			
Past due accounts: Dues	$1,735.10		
Subscriptions	1,880.30		
Sale of back numbers	973.55		
	$4,588.95		
Less 100% allowance for bad debts	4,588.95		
Current account for dues		$1,070.00	
Sundry debtors		18.30	1,088.30
Inventory of back numbers of ECONOMETRICA, at marginal cost			2,785.16
			$ 7,236.29

Liabilities

Accounts payable		$ 899.89
Accrued publishing expense:		
July-October, 1949, issue	$1,898.40	
Washington Proceedings Supplement	2,727.94	4,626.34
Dues collected in advance		154.30
Subscriptions collected in advance		458.40
Surplus: Balance, October 1, 1948	$3,613.70	
Less net loss	2,516.34	
Surplus, September 30, 1949		1,097.36
		$ 7,236.29

Summary of Surplus Account, 1931–1949

Total donations received, 1931–1949		$23,918.50
Add: Inventory of back numbers of ECONOMETRICA at marginal cost, first counted as an asset in 1945		2,489.30
		$26,407.80
Less: Grant made to International Statistical Conferences in 1947	$1,000.00	
Net deficit, 1931–1949	24,310.44	25,310.44
Surplus, September 30, 1949		$ 1,097.36



ALFRED COWLES, *Treasurer*

NOTICE OF CHANGES IN DUES AND SUBSCRIPTION RATES

The dues of members of the Econometric Society who are residents of countries other than the United States and Canada were reduced from $6.00 to $4.50 by vote of the Council of the Econometric Society in December, 1949. This action was taken in order to help offset the increase in the cost of membership (in terms of local currencies) which occurred as a result of the recent devaluation of currencies with respect to the American dollar. The new rate applies to the current 1950 dues, bills for which will be mailed in January.

The dues of members residing in the United States and Canada remain at $8.00. The subscription rate for nonmembers remains at $9.00 for all countries.

The subscription rate for graduate students recommended by a faculty member was increased from $3.00 to $4.00 by vote of the Council, effective January 1, 1950. The Council also voted to increase, as of the same date, the family membership rate from $2.00 to $3.00. The latter rate applies to one of two members of the Econometric Society who are husband and wife and who elect to receive between them only one subscription to ECONOMETRICA. Both increases were approved in order to bring these rates into accord with the increase in regular dues and subscription rates voted by the Council in 1947.

ELECTION OF FELLOWS, 1949

The Fellows of the Econometric Society have elected seven new Fellows, whose names and partial bibliographies are given below. A complete list of Fellows as of January, 1950, follows.

MAURICE ALLAIS, Institute de Statistique de l'Université de Paris, Paris, France.

A la Recherche d'une Discipline Economique, Première Partie—L'Economie Pure et Annexes, Paris: Ateliers Industria, 1943, 920 pp.

Economie Pure et Rendement Social, Paris: Imprimerie Nationale, 1945, 72 pp.

Abondance ou Misère: Propositions hétérodoxes pour un redressement de l'économie française, Paris: Librairie de Médicis, 1946, 131 pp.

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"Organisation Concurrentielle ou Planisme Autoritaire," *Mémoire des Ingénieurs Civils de France*, Vol. 99, December, 1946, pp 399–414.

"L'Oeuvre d'Irving Fisher," *Revue d'Economie Politique*, Vol. 57, May–June, 1947, pp. 459–468.

"Le Problème de la Planification dans une Economie Collectiviste," *Kyklos*, Vol. 1, No. 3, 1947, pp. 254–280, and Vol. 2, No. 1, 1948, p. 48–71

"Le Problème des Salaires," *Conférence faite dans le cadre du 25e cycle d'Etudes de la Cegos*, Paris, November, 1947, pp. 6–20.

Economie et Intérêt, Paris: Imprimerie Nationale, Vols. I and II, 1947, 800 pp.

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"Le Problème de la Coordination des Transports et la Théorie Economique," *Revue d'Economie Politique*, Vol. 58, March–April, 1948, pp. 212–271.

"Les Problèmes Economiques et Sociaux de l'Heure et leurs Solutions," *Bulletin des Transports et du Commerce*, Paris, November, 1948, pp. 690–697.

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"Productivités, Salaires Réels et Union Economique," *Economia Internazionale*, Vol. 2, August, 1949, pp. 615–629.

"Période de Production, Répartition des Facteurs Primaires de Production et Taux d'Intérêt," *Journal de la Société de Statistique de Paris*, Vol. 90, 1949.

"Rendement Social et Productivité Sociale," *Metroeconomica*, Vol. 1, No. 2, 1949.

ABRAM BERGSON, Graduate Faculty of Political Science and Russian Institute, Columbia University, New York, New York, U. S. A.

"Real Income, Expenditure Proportionality and Frisch's New Method of Measuring Marginal Utility," *Review of Economic Studies*, Vol. 4, October, 1936, pp. 33–52.

"A Reformulation of Certain Aspects of Welfare Economics," *Quarterly Journal of Economics*, Vol. 52, February, 1938, pp. 310–334.

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BERNARD CHAIT, Anvers, Belgium.

"Note sur les Systèmes Macrodynamiques," ECONOMETRICA, Vol. 3, October, 1935, p. 472.

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MILTON FRIEDMAN, The University of Chicago, Chicago, Illinois, U. S. A.

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M. G. KENDALL, London School of Economics, London, England.

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P. C. MAHALANOBIS, Indian Statistical Institute, Presidency College, Calcutta, India.

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"On the Exact Test of Association Between the Occurrence of Thunderstorms and an Abnormal Ionization" (with Subhendu Sekhar Bose), *Sankhyā*, Vol. 3, May, 1938, pp. 249–252.

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"A Review of the Application of Statistical Theory to Agricultural Field Experiments in India (Revised)," *The Indian Journal of Agricultural Science*, Vol. 10, April, 1940, pp. 192–212.

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FRANCO MODIGLIANI, Bureau of Economics and Business Research, University of Illinois, Urbana, Illinois, U. S. A.

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"Fluctuations in the Saving-Income Ratio: A Problem in Economic Forecasting," Conference on Research in Income and Wealth, *Studies in Income and Wealth*, Vol. 11, New York: National Bureau of Economic Research, 1949, pp. 371–443.

PROPOSED ELECTION OF FELLOWS

The Fellows of the Econometric Society will elect new Fellows in 1950. Only Fellows are entitled to vote in this election, but all members of the Society are entitled to suggest names of candidates. These will be presented to the Council for such action as it sees fit when it makes nominations for Fellowships. Suggestions from members should be sent to William B. Simpson, Secretary, The Econometric Society, The University of Chicago, Chicago 37, Illinois, U. S. A., to reach him before April 15, 1950.

A list of the present Fellows is given on pages 98–99 of the present issue.

FELLOWS OF THE ECONOMETRIC SOCIETY

JANUARY 1950

Professor MAURICE ALLAIS, Paris, France.
Professor R. G. D. ALLEN, London, England.
Professor LUIGI AMOROSO, Rome, Italy.
Professor OSKAR N. ANDERSON, Munich, Germany.
Professor ABRAM BERGSON, New York, New York, U. S. A.
Professor ARTHUR L. BOWLEY, Haslemere, Surrey, England.
Professor COSTANTINO BRESCIANI-TURRONI, Milan, Italy.
Professor ARTHUR F. BURNS, New York, New York, U. S. A.
Mr. BERNARD CHAIT, Anvers, Belgium.
Mr. COLIN CLARK, Brisbane, Australia.
Professor J. M. CLARK, New York, New York, U. S. A.
Mr. ALFRED COWLES, Chicago, Illinois, U. S. A.
Professor WILLIAM LEONARD CRUM, Berkeley, California, U. S. A.
Professor HAROLD T. DAVIS, Evanston, Illinois, U. S. A.
Professor GUSTAVO DEL VECCHIO, Rome, Italy.
Dr. J. B. D. DERKSEN, Lake Success, New York, U. S. A.
Professor FRANÇOIS DIVISIA, Paris, France.
Dr. PAUL H. DOUGLAS, Washington, D. C., U. S. A.
Dr. LUIGI EINAUDI, Rome, Italy.
Professor GRIFFITH C. EVANS, Berkeley, California, U. S. A.
Dr. MORDECAI EZEKIEL, Washington, D. C., U. S. A.
Professor MILTON FRIEDMAN, Chicago, Illinois, U. S. A.
Professor RAGNAR FRISCH, Oslo, Norway.
Dr. ROBERT GIBRAT, Paris, France.
Professor CORRADO GINI, Rome, Italy.
Professor TRYGVE HAAVELMO, Oslo, Norway.
Professor GOTTFRIED HABERLER, Cambridge, Massachusetts, U. S. A.
Professor FRIEDRICH A. HAYEK, London, England.
Professor J. R. HICKS, Oxford, England.
Professor HAROLD HOTELLING, Chapel Hill, North Carolina, U. S. A.
Professor LEONID HURWICZ, Urbana, Illinois, U. S. A.
Dr. NICHOLAS KALDOR, London, England.
Dr. M. KALECKI, Lake Success, New York, U. S. A.
Mr. M. G. KENDALL, London, England.
Dr. LAWRENCE R. KLEIN, New York, New York, U. S. A.
Dr. N. D. KONDRATIEFF, U. S. S. R.
Professor TJALLING C. KOOPMANS, Chicago, Illinois, U. S. A.
Professor SIMON KUZNETS, Philadelphia, Pennsylvania, U. S. A.
Professor OSCAR LANGE, Cracow, Poland.
Professor WASSILY LEONTIEF, Cambridge, Massachusetts, U. S. A.

Professor Abba P. Lerner, Chicago, Illinois, U. S. A.
Professor Erik Lindahl, Uppsala, Sweden.
Professor P. C. Mahalanobis, Calcutta, India.
Professor Jacob Marschak, Chicago, Illinois, U. S. A.
Professor Lloyd A. Metzler, Chicago, Illinois, U. S. A.
Professor Frederick C. Mills, New York, New York, U. S. A.
Professor Franco Modigliani, Urbana, Illinois, U. S. A.
Professor Henry L. Moore, Cornwall, New York, U. S. A.
Professor Giorgio Mortara, Rio de Janeiro, Brazil.
Dr. Jacob L. Mosak, Lake Success, New York, U. S. A.
Professor Gunnar Myrdal, Stockholm, Sweden.
Professor Bertil Ohlin, Stockholm, Sweden.
Professor Egon S. Pearson, London, England.
Dr. Charles F. Roos, New York, New York, U. S. A.
Professor René Roy, Paris, France.
Professor Jacques Rueff, Paris, France.
Professor Paul A. Samuelson, Cambridge, Massachustts, U. S. A.
Professor Erich Schneider, Kiel, Germany.
Professor Joseph A. Schumpeter, Cambridge, Massachusetts, U. S. A.
Dr. Walter A. Shewhart, New York, New York, U. S. A.
Dr. Arthur Smithies, Washington, D. C., U. S. A.
Dr. Hans Staehle, Geneva, Switzerland.
Mr. John Richard Stone, Cambridge, England.
Professor Jan Tinbergen, The Hague, The Netherlands.
Professor Gerhard Tintner, Ames, Iowa, U. S. A.
Professor Felice Vinci, Milan, Italy.
Professor John von Neumann, Princeton, New Jersey, U. S. A.
Professor Abraham Wald, New York, New York, U. S. A.
Dr. Frederick V. Waugh, Washington, D. C., U. S. A.
Professor Samuel S. Wilks, Princeton, New Jersey, U. S. A.
Professor Edwin B. Wilson, Boston, Massachusetts, U. S. A.
Professor Herman O. A. Wold, Uppsala, Sweden.
Professor Elmer J. Working, Urbana, Illinois, U. S. A.
Professor Holbrook Working, Stanford, California, U. S. A.
Dr. Theodore O. Yntema, Dearborn, Michigan, U. S. A.
Professor F. Zeuthen, Rungsted, Denmark.

MEETINGS OF THE SOCIETY IN 1950

Tentative plans call for one or more meetings of the Econometric Society to be held in Europe during 1950, as well as for the initiation of regular regional meetings in India and Japan.

Three meetings are scheduled for the United States in 1950: a regional West Coast meeting in Los Angeles, California, in June (see announcement below); a meeting at Harvard University in Cambridge, Massachusetts, during the first week in September, concurrent with the International Congress of Mathematicians; and a meeting in Chicago, Illinois, in late December in conjunction with meetings of the American Economic Association, American Statistical Association, Institute of Mathematical Statistics, and related social science organizations.

Further information regarding these meetings will appear in the April issue of ECONOMETRICA.

ANNOUNCEMENT OF LOS ANGELES MEETING JUNE, 1950

Starting with a joint meeting in mid-June, 1950, with the Institute of Mathematical Statistics at the University of California at Los Angeles, the Econometric Society plans to initiate as a regular practice the holding of an annual regional meeting of the Society on the West Coast. The program will include sessions with emphasis on economic theory as well as sessions organized jointly with the Institute. Although definite dates have not been set for the meeting, it is expected that the Econometric Society sessions will start on Wednesday or Thursday, June 14 or 15, and continue for three or four days, ending in time to permit attendance at the summer session on mathematical statistics to be held on the Berkeley campus of the University of California.

Abstracts of papers intended for presentation at the Los Angeles meeting and other correspondence regarding the meeting should be addressed to the program chairman, Professor Kenneth J. Arrow, Department of Economics, Stanford University, Stanford, California. Other members of the program committee include Stephen Enke, The Rand Corporation and University of California at Los Angeles; Frank L. Griffin, Reed College; Ernest S. Keeping, University of Alberta; George M. Kuznets, University of California, Berkeley; George A. Lundberg, University of Washington; Verne S. Myers, Lockheed Aircraft Corporation; Laszlow Radvanyi, National University of Mexico; Kenneth D. Roose, University of California at Los Angeles; and William B. Simpson, Cowles Commission for Research in Economics (ex officio).

Further details as to arrangements will be contained in the April issue of ECONOMETRICA and in a preliminary program which will be mailed American members in advance of the meeting.

EDITOR'S NOTE

Publication of the Supplement to ECONOMETRICA containing the econometric papers presented at the Washington meeting of the Econometric Society, held in conjunction with the International Statistical Conferences, September 6–18, 1947, has been delayed due to unforeseen difficulties. The Supplement is part of Vol. 17 of ECONOMETRICA, and will be distributed in the spring of 1950 to those who were members or subscribers in 1949.

The report of the Boulder meeting of the Econometric Society, held August 29–September 1, 1949, will appear in the April, 1950, issue together with the report on the symposium on mathematical training of social scientists. The report of the New York meeting will appear in the subsequent issue.

CRITICISM INVITED

LETTER TO THE EDITOR

DEAR SIR:

The entropy–ophelimity analogy of Lisman (ECONOMETRICA, Vol. 17, January, 1949, pp. 59–62) is supposed to illustrate the division of income into expenditure and saving as an outgrowth of the principle of maximization. I believe the mathematical part of this analogy can be further specified and then applied in the general case, namely when heat (energy) is distributed among n heat containers, as is income distributed among n budgetary columns. In my "Dynamical Theories of Money and the Mechanical Analogies" (*Kozgazdasági Szemle*, No. 1-2, 1933, Budapest) I made the following comparisons: (a) stock of energy Q corresponds to stock of commodity (money) m, (b) reciprocal of temperature $1/T$ corresponds to marginal utility u, and (c) entropy ζ corresponds to ophelimity U.

The reciprocal of the temperature (considered a function of quantity of heat and of specific heat) is treated in thermodynamics often as a separate function under the name "reduced quantity of heat." (Clausius.) It is understood that, specific heat taken constant, this function has the shape of a hyperbola. Its integral (entropy) is therefore a logarithmic function of the quantity of heat. Accordingly, we have a hyperbola for marginal utility and a logarithmic curve for ophelimity. The analogy is reflected by the following two integral formulas: $\zeta = \int dQ/T$ for entropy and $U = \int u\, dm$ for ophelimity. The analogy allows the discussion of the equilibrium situations on parallel lines. In a set of n equations the first equation expresses *conservation of stock* (heat, commodity). The additional $(n - 1)$ equations cover the marginal principle of maximization, translating equality of *marginal entropies* over n heat containers and equality of *marginal utilities* over n budgetary columns.

At first sight Lisman's statistical interpretation of the entropy analogy in view of monetary phenomena appears to be promising. A similar interpretation seems to have been started by Divisia in 1924 and Rueff in 1927 and was continued in my kinetic theory of money (1932–35). The ultimate interpretation is still open and econometricians look forward with interest to Dr. Lisman's book.

Chicago, Illinois ANDREW PIKLER

ECONOMETRICA

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JOSEPH A. SCHUMPETER

(1883–1950)

By WASSILY LEONTIEF

"INVESTIGATORS existed in all times who stood on a height from which they were able to survey all the land around them rather than a single valley, who not only superficially and from a purely formal point of view recognized the right to exist of the various schools of thought—that amounts to very little—but who had a real understanding of their ideas and felt emotional affinity toward all of them."[1]

Schumpeter was one of these exceptional men. The fortunes of a rich and varied personal experience were favorable to him. The son of a well-situated upper-class family, he spent the early part of his life in Vienna, one of the most sheltered and pleasant valleys of the old world. Still, before 1914 he had an opportunity to learn to know England, to spend two years in the peculiar colonial atmosphere of Cairo, and even to visit the United States.

Lest we mistake the nature of these experiences, let us remember that young Schumpeter saw the prewar world not as a member of an expeditionary force, as a refugee, neither as an official representative nor negotiator of some kind or another, not even as a vacationing tourist. He saw it truly as "a gentleman and a scholar." Hunting meadows rather than green conference tables, scientific or in any case intellectual conversations rather than diplomatic wrangling marked his early cosmopolitan experience. He knew that old world at its best and he did not want to see a different one. He never visited Europe after 1937. One is inclined even to interpret Schumpeter's brief sojourn in Japan and his final move to the United States as an attempt to escape the uncongenial social and intellectual climate of the post-First World War Europe. In the meantime he had his brief stretch of active political experience as Minister of Finance in an early republican government of the partitioned Austria; an experience which must have only reinforced his often-expressed conviction that science and politics, the urge to

[1] Translated from Joseph A Schumpeter, "Vergangenheit und Zukunft der Sozialwissenschaften," *Schriften der Sozialwissenschaftlichen Academischen Vereins in Czernowitz*, Munchen und Leipzig, 1915, p. 105

Editor's Note Acknowledgement is made to the American Economic Association for permission to use the photograph on the preceding page.

understand and the drive to act, far from being complementary, are actually incompatible.

Unlike Lord Keynes who was born in the same year, 1883, but whose intellectual progress like that of a mountain path was slow at the beginning and reached its highest point in all but vertical ascent of the final stretch, Schumpeter attained his full scientific stature very early, before the age of thirty. Thus he had the unique experience of enjoying personal contact with Walras, Bœhm-Bawerk, and the other great of what now appears to be a bygone era, on the one hand, and taking part in the development of some of the most recent movements in our science on the other.

One should not, however, overestimate the role that a man's direct personal experience can play in the formation of his intellectual make-up. Such a relation frequently exists, and in the case of Schumpeter it definitely does, but it is of a more subtle, indirect kind. The strongest single impression with which one was left after having spent an hour with him in the classroom or at a scientific meeting, or even better on a leisurely walk along the wooded shores of the lake near his Taconic, Connecticut, country home, was that of the astounding width of Schumpeter's intellectual horizon. He was equally at home in early Greek philosophy, English parliamentary history, Italian literature, and French romanesque architecture. One wonders whether, if he had pushed his roots in one single country, Schumpeter would have still become the spiritual citizen of the world that he was, at least in the latter part of his life. But again the connection between cause and effect might have actually worked in the opposite direction.

To economics and to the social sciences in general Schumpeter brought a remarkable combination of imaginative vision and uncommon critical acumen. The vision led him to the formulation of his well-known notion of economic development. However significant, that contribution alone cannot account for the influence which Schumpeter exerted on the fortunes of our discipline during the last four decades. What was it then that made his position so prominent and his departure now so keenly felt?

It was Schumpeter's peculiar concern with the science, with the process of knowledge itself. The intellectual adventure meant more to him than the ultimate discovery. To prepare the ground for new ideas, to watch and to protect their growth, gave, I suspect, more satisfaction to him than even the final harvest; and he was wise enough—some will say skeptical enough—to realize that in the fields of intellectual endeavor no harvest is final, all fruit is perishable and only as good as the new seed it might contain.

This essentially methodological approach to scientific knowledge in general and economics in particular determines the whole tenor of his

first major work, *Das Wesen und der Hauptinhalt der theoretischen Nationalokonomie* (Munchen und Leipzig: Dunker und Humblot, 1908). Never translated or even republished, this remarkable book remains practically unknown in the English-speaking world and yet it contains the statement of his fundamental views which constitute the basis of Schumpeter's whole scientific *weltanschaung*. Some of these were never restated again as explicitly or with so much *elan*. It is indicative of his turn of mind that the nearest approximation to exposition of the general principles of economics was undertaken by Schumpeter at the very beginning of his career. In a sense it is an inventory, but certainly not a stock-taking of finished products counted at the end of a working day. It is rather a survey of tools and materials, a preliminary pilot study of the working processes.

A very sharp distinction is being drawn between description and explanation, between explanation and the so-called practical conclusions, i.e., political recommendations. Schumpeter's thinking on the role and the internal structure of economic theory was admittedly shaped by his early study of the founders of the mathematical school: Walras, who in Schumpeter's view stood all by himself far above the rest of them, Pareto, Cournot, Barone, Auspitz, and Lieben. All these names and some others are already mentioned in his first published article which appeared in 1906 under the title, "Ueber die mathematische Methode in der theoretischen Nationalökonomie."[2] The insistent emphasis on the formal nature of all theoretical thinking, a vigorous showing-up of all kinds of terminological humbug underlying much of the apparent conflict between supposedly incompatible theories, the basic methodological distinction between data and variables, a masterly exposition of the "method of variation"—later popularized under the name of "comparative statics"—reveals a level of methodological self-consciousness hardly yet reached by most economic writing up to the present time. If it were, much of the pro- and anti-Keynesian literature of the later thirties and early forties would not have seen the light of day.

The insight into the nature and appreciation of the true significance of a theory or a scientific procedure often finds its most sensitive measure in a clear and unequivocal statement of its inherent limitations. While extolling the elegant precision and extraordinary hitting power of the pure, essentially mathematical, economic theory, Schumpeter had already in this, his first major work, delineated the margins of its effective range. He specifically designated what he called the process of development as the particular aspect of economic reality which could not be encompassed by the conceptual schemes of static general equilibrium theory.

[2] *Zeitschrift fur Volkswirtschaft, Sozialpolitik und Verwaltung,* Vol. 15.

The question was asked. The answer came less than four years later with the publication of the *Theorie der wirtschaftlichen Entwicklung* (Leipzig: Dunker and Humblot, 1912).

Of all Schumpeter's writings this has been and, as one might expect, will also in the future be referred to as that which contains his most important contribution to economics. The dramatic concept, it is possibly more appropriate to use in this context the word idea, of the creative entrepreneur, the "new combination," the distinction between the operation of disequilibrating forces and the subsequent process of adjustment toward equilibrium are too well-known to be dwelt upon in these few pages.

Those who find pleasure, as Schumpeter himself did, in tracing devious affiliations of ideas should note the remarkable affinity which exists between Schumpeter's economic development and Henry Bergson's equally famous *creative evolution* (the book of the same name first published in 1907). Here, as there, spontaneous variation is sharply contrasted with "mechanical" repetition, unpredictable change and innovation with ordinary imitation. The common basis of the two conceptual schemes is even more strikingly revealed in the similarity of the gnosiological implications. The philosopher and the economist alike concluded that creative development lies outside the legitimate domain of mathematically formulated "static" theory.

So far as the exact position of the line of demarcation between the "static" and the developmental phenomena is concerned, Schumpeter's position seems to have changed gradually, reflecting at each stage the interim advance of the theoretical analysis itself. Already in his first book and the original, 1910 article on Economic Crises ("Ueber das Wesen der Wirtschaftskrisen," *Zeitschrift für Volkswirtschaft, Sozialpolitik und Verwaltung*, Vol. 19) the process of ordinary population growth and of the corresponding capital accumulation were interpreted as essentially static. By 1939, when the two volumes of his *Business Cycles* came out, Tinbergen's, Frisch's, Kalecki's, and many other dynamic models had shown oscillatory processes to lie entirely within the reach of "mechanical" explanatory schemes. While fully appreciating the importance of that great theoretical advance, Schumpeter considered it to represent a necessary expansion of what he originally called the "static" analysis. The essentially nonautomatic, unique and, because of that, theoretically unpredictable features of economic development retained in his eyes an important and even predominant role in the explanation of the ups and downs of what he called "the economic process of the capitalist area."

The spectacular progress of dynamic economics witnessed during the recent years might lead us to believe that the extension of the realm of

mechanical models will go apace. But again one must admit that the fundamental problem of technological change has hardly yielded an inch to that powerful surge. Of all nonstatic processes the progress of the science itself is possibly the least predictable.

When published in 1908, Schumpeter's theory of interest was first of all a bold declaration of independence. When four years later in his *Theory of Economic Development* the "spirited young author," instead of mending his ways, reiterated the assertion that the rate of interest would necessarily be zero in a static economy, Bœhm-Bawerk himself, in grief rather than scorn, published a sixty-page long refutation. Schumpeter, in his forty-page retort was respectful but unyielding. Bœhm-Bawerk closed the exchange with a twenty-page "final remark."[3]

From then on up to the present time, Schumpeter's theory of interest was mentioned in the literature often enough but nearly always with the embarrassed condescension with which one refers to an admittedly inferior work of an otherwise excellent artist. A melancholy recognition of this situation transpires in his own treatment of the subject in the *Business Cycles.* Although steadfastly maintaining his original position, the author "endeavored throughout to formulate the propositions in this book in such a way as to make them, whenever possible, acceptable also to those who differ from him in their view as to the nature of interest."[4]

Rereading now the original controversy, one cannot but observe that in the light of subsequent experience the most important second and third of Bœhm-Bawerk's well-known "three grounds" can certainly be subject to reasonable doubt. Is his famous psychological law of "perspective underestimation of future needs" really as universal as the author of *The Positive Theory of Capital* would like us to believe? Without shifting to the opposite extreme of the Keynesian position, one certainly must admit that there do exist "psychological" reasons for accumulation other than the attraction of a positive rate of interest, for example, the natural desire to provide for one's old age. Moreover, even if every individual consumed by the time of his death all savings accumulated during the early part of his life, the society as a whole—even in the case of perfectly stationary population—would obviously maintain a certain positive amount of capital assets (even at a zero rate of interest). Had the marginal productivity of capital—Bœhm-Bawerk's "third ground"—been positive whatever its total amount, the equilibrium rate of interest would still of course be necessarily greater

[3] All three articles in *Zeitschrift für Volkswirtschaft, Sozialpolitik und Verwaltung*, Vol. 22, 1913, pp. 1, 599, and 640.

[4] *Business Cycles*, Vol. 1, New York and London: McGraw-Hill Book Co., 1939, p. 123n.

than zero. This Schumpeter has never denied. He did contest, however, his opponent's factual assumption that with a given amount of land and labor and a given technological horizon, e.g., without new inventions, any amount of capital, however large it might be, would necessarily have positive marginal productivity. He thought it more realistic to admit the possibility of a situation in which an addition to the already available stock of capital would not be able to increase the net output. If the size of that stock happens to be as large as, or at least not larger than, the amount that the society is ready to hold even without a positive rate of interest, the model of a static economy with a zero rate of interest is complete. Schumpeter might not have been so wrong after all.

Turning from the analysis of the present economic system to the question of its future—to the problem of socialism—Schumpeter, as he did already on previous occasions in his essays on the sociology of imperialism,[5] and on the theory of social classes,[6] went far beyond the limits of technical economics. In contrast to many of his professional colleagues he long ago (I am told, as early as 1919) had taken the by now generally accepted position that from a purely technical point of view a socialist economy is entirely possible. His prediction of the inevitable demise of the area of private enterprise is based, however, on a broad and penetrating analysis of the inner workings of our whole social and political system. Nothing reflects the intellectual stature of the man better than the fact that he made this prediction while despising the entire range of social and cultural values which the new society, according to his analytical conclusions, would necessarily inaugurate.

Though his principal lasting contributions were made outside the specialized field of mathematical economics and econometrics, Schumpeter had a profound understanding of its aims and methods and an unbounded appreciation of its achievements. While Marshall, although he is known to have made extensive use of the tools of mathematical analysis, was obviously at great pains to conceal that fact, Schumpeter, who hardly ever resorted to mathematics in his original work, was ever ready to pay highest tribute to the queen of sciences. As it is always in the case of a real theorist, the esthetic element played a considerable part in that appreciation. Thus it should not be considered flippant to compare Schumpeter's relation to mathematical economics with a true balletomane's attitude toward the classical ballet. He was not an active practitioner of the high art himself—none of his writings contains even a single mathematical argument of any length; he was able, however,

[5] *Zur Soziologie der Imperialismen*, Tubingen, 1919.

[6] "Die sozialen Klassen im ethnisch homogenen Milieu," *Archiv für Sozialwissenschaft und Sozialpolitik*, Vol. 57, 1927.

to appreciate its fine nuances, knew its history, was intimately familiar with its present state and vitally interested in its future.

Nothing was "farther from [his] mind than any acrimonious belief in the exclusive excellence of mathematical methods,"[7] and he would have been the first to admit that a most exceptional mind should be able to find its way through the maze of an intricate theoretical problem without recourse to formal devices of mathematical analysis. He equally well knew, however—what many of his professional colleagues unfortunately do not—that an ordinary first class mind, if not so equipped, is more likely than not to fall prey to logical pitfalls lurking around the corners of any extended theoretical argument.

As one of the founders and the fifth president (1940–41) of the Econometric Society, as a writer and teacher, Schumpeter was to a large extent responsible for the ever wider acceptance of the view that mathematical methods represent one of the most useful, well-nigh indispensable, instruments in the tool box of a modern economist.

While most economists, including those in the first ranks of the profession, devote today much of their time and energy to the practical task of oiling the creaking wheels of our economic machine, strain their shoulder in helping to heave it out of the rut of whatever current crises it finds itself in, or at least join the circle of eager onlookers supplying gratuitous advice from the sidelines, Schumpeter shied away from the practical attitude. By conviction, as well as temperament, he was a thinker rather than a doer.

While others are preoccupied with worries about the future of the economic system and our society, he was equally concerned with the future of the economic and social science. It is only natural that he should have developed a lasting interest also in its past. The methodological orientation of his general point of view combined with constant preoccupation with questions of change and development quite early turned Schumpeter's attention toward the history of the economic doctrine. His "Epochen der Dogmen- und Methodengeschichte"[8] came out in 1914. A year later, so to speak still in the same breath, the little-known pamphlet, "Vergangenheit und Zukunft der Sozialwissenschaften" was published (from which the opening quotation of this paper stems). Work on more current issues had relegated, then, this particular interest to the background where it remained for over thirty years. Only the series of brilliant and penetrating memorial essays, beginning in significant sequence with Walras and ending with the posthumously published article on Mitchell, shows how alive it actually

[7] J. A. Schumpeter, "The Common Sense of Econometrics," ECONOMETRICA, Vol. 1, January, 1933, pp. 5–12.

[8] *Grundriss der Sozialökonomie*, Tubingen, Vol. 1, Part 1.

was all that time. In 1943 Schumpeter began the work on a large history of economic analysis. By the time of his death the principal portions of the manuscript were substantially completed.

If his refusal to be preoccupied with the so-called immediate practical problems of the day, to take a public stand on issues raised by a most recent economic message of the President, or to contribute to a symposium on the latest monetary stabilization scheme is to be taken as a sign of his retirement to the ivory tower, Schumpeter had lived in such retirement for at least the last fifteen years.

But what an unusual ivory tower it was! Set up in the very middle of the intellectual traffic of our time, it was ever full of visitors. Everyone was invited to enter its wide-open doors, everyone, that is, who had an idea to discuss, be it in economics or sociology, in history or art. An undergraduate student with a low record but a hunch about a possible new solution of the duopoly problem would be not less and—if the hunch appeared to be good—would be even more welcome than a prominent visiting scholar. Both would be met with lively interest and would leave stimulated and encouraged. Even personal woes were somehow elevated to the intellectual sphere and advanced toward an objective solution if such were at all possible. A pessimist and a skeptic in his view on the future of our western civilization which he cherished so much, Schumpeter was an optimist in his belief in the boundless progress of the inquiring mind.

Harvard University

RATIONAL BEHAVIOR, UNCERTAIN PROSPECTS, AND MEASURABLE UTILITY[1]

By Jacob Marschak

After introducing some basic concepts and three postulates on rational choice, it is proposed to show that if the economists' theory of assets is completed by a fourth postulate on rational choice, then utility can be defined as a quantity whose mathematical expectation is maximized by the rational man. In this sense, utility is "measurable" and "manageable." These results are inspired by von Neumann's and Morgenstern's discussion of utility in *Theory of Games and Economic Behavior;* an attempt is made to sketch some relations between their approach and the present one. It is shown in conclusion that while gambling is compatible with the four postulates, the "love of danger" is not; and a property of the maximum mathematical expectation of utility is conjectured.

I. SOME BASIC CONCEPTS

1. *Rational behavior: description and advice.* The theory of rational behavior is a set of propositions that can be regarded either as idealized approximations to the actual behavior of men or as recommendations to be followed.

This can be shown in the following proposition: "The rational man does not make logical and arithmetical errors." Or, to give three particular examples: "if $x = 2$ and $y = 0.005$, the rational man concludes that $xy = 0.01$"; "if all A are B, the rational man concludes that all non-B are non-A, but he does not conclude that all B are A"; "if P follows from Q, he concludes that non-Q follows from non-P, but he does not conclude that Q follows from P." Now, a large proportion of people, especially when in a hurry to answer and to act, are apt to disobey those rules. Living men and women are apt to misplace their decimal points; they often conclude from the part to the whole; and they are not above confusing a sufficient condition with a necessary one. That is, they do not behave rationally. What is, then, the use of setting up the above propositions of arithmetic and logic? The use is twofold: to describe approximately the behavior of men who, it is believed, cannot be "all fools all the time," and to give advice on how to reach "correct" con-

[1] Based on a Cowles Commission Discussion Paper, Economics No. 226 (hectographed), July, 1948, and a paper presented at the Madison Meeting of the Econometric Society, September, 1948, Marschak [4]. This article will be included in Cowles Commission Paper, New Series, No. 43.

Acknowledgments are due to various present and former members of the Cowles Commission staff, especially Kenneth J. Arrow, Herman Chernoff, Nathan J. Divinsky, Clifford Hildreth, and Herman Rubin; to Leonard J. Savage, The University of Chicago; and to Abraham Wald, Columbia University.

clusions. These two aspects of the rules of logic and arithmetic can be called, respectively, the descriptive and the recommendatory aspect.

As an approximative description of actual behavior, the rules of logic and arithmetic may be regarded as hypotheses on the psychology of reasoning and calculating: hypotheses whose predictive power ("predict how many people of a certain age-group, etc., will make a certain kind of error") is susceptible of empirical tests. Viewed from their recommendatory aspect, as advice on how to reach "correct" conclusions, the rules of logic and arithmetic are susceptible only to the test of internal consistency; in fact, the aggregate of these rules (including the rule of internal consistency itself) can be regarded as nothing but the definition of "correctness" of conclusions.

Note, however, that "correct" conclusions are often regarded as having, in addition, a "utilitarian" virtue. The results of decisions based on "correct" conclusions are, in some sense, "preferable" to results of decisions based on incorrect ones. It is "advisable" to follow the rules of logic and arithmetic. In dealing with his environment ("nature," which includes "society") a man who often makes mistakes in his inferences and his sums is, in the long run, apt to fare less well than if he had been a better logician and arithmetician. Thus concepts of "welfare" and of "long run," or of utility and mathematical expectation, become involved. If advisable decisions must obey rules of logic and arithmetic, they are not themselves defined by those rules alone. The fulfillment of rules of conventional logic and arithmetic is a necessary but not a sufficient condition for a decision to be advisable. We need additional definitions and postulated rules, to "prolong" logic and arithmetic into the realm of decision. We shall define rational behavior as that which follows those rules, in addition to the rules of logic and arithmetic. It will be seen (an experience common in logic and mathematics) that alternative sets of postulated rules may or may not lead to different implications.

Again (as in the case of the logico-arithmetical rules in the narrower sense) these postulates, and their implications, can be used in two ways: they can be conceived as approximate descriptions of actual behavior or they can be regarded as rules of behavior to be followed. Both the descriptive and the recommendatory aspects have practical use. For example, a government (or a firm) has to (a) choose the advisable course of its own responses to environment and (b) know, as a part of its environment, something about the actual behavior of tax-payers, consumers, competitors, etc.

Rational behavior is also called the behavior of a rational man. For brevity, we shall speak of "the man," to cover the rational consumer, the rational firm, the rational government, etc.

2. *Complete information.* We have further to distinguish between two

cases: the case when the man thinks he knows certain relevant probability distributions, and the case when the man does not think so. We call the former case *complete information*, and the latter, *incomplete information.*[2] The theory of rational behavior under incomplete information is not presented here. It would be related to the studies on the rationale of sampling, started by Neyman and Pearson [1] and developed particularly by A. Wald, e.g., [1, 2]. The case of complete information can (but need not) be represented as a limiting one: it is approached as the number of observations which have been made by the man increases. Alternatively, one can think of the probabilities used in the man's decisions as "degrees of belief" not related to specific samples in any simple way, yet obeying the usual axioms on probabilities.

A special case of complete information is that of *certainty*: in this case all probabilities have values 0 or 1.

3. *Future histories x; commodity space X.* At a given point of time the man considers a sequence of, say, τ future time intervals, up to a certain time point called horizon. The amounts of each commodity consumed during each interval and the stocks of various kinds available at the horizon point define one of the mutually exclusive "future histories." Each such history can be represented by a point x in the "commodity space" X. Since, for the more remote future, the man may have a coarser classification of goods than for the immediate future (a point emphasized by Koopmans [1]), some commodities, or commodity groups, may have to be entered identically as zeros for some time intervals or for the horizon point; hence the dimensionality of X at most equals $\gamma(\tau + 1)$, where γ is the number of commodities ever to be entered.

One can also interpret each point x more narrowly, as the development of a particular "venture" ("play"), disregarding all other aspects of a man's future. Thus, a firm owner's choice between various uses of cash profits in his personal plans of consumption may be considered irrelevant (this diminishes γ); only the near future is considered (this makes τ small); and attention is paid only to a particular shop or ship, patent or contract. Note, however, that even if one is concerned only with a single money wager, the number of alternative points x (here the alternative amounts of money gains) may be large.

Inasmuch as the man may treat time and the amounts of goods as both discrete and bounded, we shall assume the number of points x finite, though, of course, as large as we like. We shall enumerate these points as $x^{(0)}, \cdots, x^{(\nu)}$.[3]

[2] Uncertainty in Knight's [1] sense is presumably identical with incomplete information. See also Marschak [5] (pp. 108–110) and [6a] (pp. 183–184, 193–195); and Hart [1, 2].

[3] The case of a possibly infinite ν has been treated by Rubin [1], with results substantially unchanged. The choice between ν finite or infinite is one of mathe-

4. *Prospects a: domain A.* We call a prospect a the vector (the distribution) of the probabilities $a_0, \cdots, a_\nu$, assigned respectively to all the mutually exclusive histories $x^{(0)}, \cdots, x^{(\nu)}$. We can write $a = a(x)$, where x is a random vector.[4] We have

$$(4.1) \qquad \sum_{\mu=0}^{\nu} a_\mu = 1, \qquad a_\mu \geqslant 0, \qquad (\mu = 0, \cdots, \nu).$$

Accordingly, we can represent each prospect by a point $a = (a_1, \cdots, a_\nu)$ in the domain A of the Euclidian ν-space defined by the condition that A consists of all points a such that for every a,

$$(4.2) \qquad \sum_{\mu=1}^{\nu} a_\mu \leqslant 1, \qquad a_\mu \geqslant 0, \qquad (\mu = 1, \cdots, \nu);$$

we define a_0:

$$(4.3) \qquad 0 \leqslant a_0 = 1 - \sum_{\mu=1}^{\nu} a_\mu \leqslant 1.$$

Note that A is an infinite set regardless of whether ν is or is not infinite, because each probability can take all real values from 0 to 1 subject only to the restriction (4.1).

If a prospect promises one particular history, say $x^{(\mu)}$, $(\mu = 0, \cdots, \nu)$, with certainty, we call it a *sure prospect* and denote it by $a^{(\mu)}$. Prospects which are not sure are called uncertain.

5. *Geometric presentation: boundary and interior of A.* In Figure 1, $\nu = 2$, and the sure prospect $a^{(0)}$ is chosen as the origin. The two other sure prospects $a^{(1)}$, $a^{(2)}$, lie on the two axes, each at unit distance from the origin. The lengths of segments $\overline{am} = a_0$, $\overline{an} = a_1$, $\overline{ap} = a_2$, indicate the component probabilities of the uncertain prospect a; and $a_0 + a_1 + a_2 = 1$.[5] All points in or on the rectangular isosceles triangle $a^{(0)}a^{(1)}a^{(2)}$ constitute the domain A, as defined by (4.2). For any ν, the sure prospect $a^{(0)} = (0, \cdots, 0)$ is at the origin; while any other sure prospect, $a^{(\mu)}$, $\mu = 1, \cdots, \nu$, will be at some other corner of the domain A; $a^{(\mu)}$ is a unit vector: each of its ν components $a_\pi^{(\mu)} (\pi = 1, \cdots, \nu)$ is

matical elegance rather than realism: sugar is retailed by the pound, interests accrue daily, and horizons are limited.

[4] We use letters $x, a, b, c, \cdots$, with or without superscripts for vectors, and $a_1, \cdots, a_\mu, \cdots, a_\nu$, for components of a. The letters $q, r, \cdots$, denote scalars. A line connecting two points is denoted thus $\overline{ab}$. Capital letters A, F, J, U, V, X, are used for sets or classes.

[5] A more symmetrical but less familiar presentation would measure a_0, a_1, a_2 as the distances of a point a from each of the sides of an equilateral triangle.

1 or 0 according as $\pi =$ or $\neq \mu$. Any prospect a for which at least one component is zero, lies on the boundary of A. Any prospect not on the boundary of A is called an interior point.

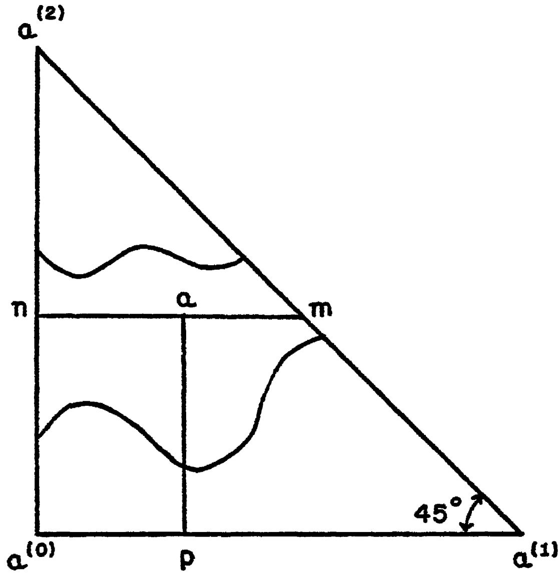

FIGURE 1

6. *The feasible set F.* The set of all prospects that the man thinks he can achieve by his actions is a subset of A. It will be called the feasible set and denoted by F. It is determined by the man's external surroundings as well as his own endowment: the technology of his farm or factory, his ability to borrow, the market conditions, etc. Set F is not dependent on the man's "tastes," to be defined in Section 8 below. In Figure 1, set F is indicated by the area bounded by, and including, the curved line and segments of axes. We shall always assume F to be a closed set.

II. THREE POSTULATES ON CHOICE

7. *Commodity space X and probability space A.* The economic theory of choice was originally applied to alternative bundles of commodities, each promised with certainty, or to alternative combinations of this year's, next year's, third year's . . . flows of money (Fisher [2]) or of some other single commodity, again promised with certainty. We have called the set of such alternatives the space X. To treat uncertain prospects we have introduced the domain A of the ν–dimensional

probability space, $\nu + 1$ being the number of points in the commodity space X. The propositions (postulates, definitions, and derived propositions) on choice, formulated in this Part for points in the probability space, hold good also for the choice between commodity bundles or sequences of such bundles,[6] provided A is replaced by X. However, the following remarks should be made:

(1) The coordinates of A are continuous probabilities; if coordinates of X represented continuous quantities of goods, ν would be infinite. But see footnote 3.

(2) While the points in X are usually (not always) required to have nonnegative coordinates, the points in A are restricted more strongly by (4.2).

(3) In addition to the postulates of this Part a postulate of "monotonicity" is occasionally proposed for the theory of choice in the commodity space X. This postulate must differ from its counterpart in A and will deserve special discussion, in Part VIII below.

(4) It will be noted that for choices in the probability space, Postulate III introducing indifference surfaces can be replaced by a weaker one, III′ or III*.

8. POSTULATE I: *Complete ordering.* One assumes a relation $\mathfrak{g}$ (read "at least as good as") between elements $a, b, c, \cdots$ of A, such that the following two postulates hold: POSTULATE Ia: *For any two a, b, at least one of the following holds*: $a \,\mathfrak{g}\, b$, $b \,\mathfrak{g}\, a$ (*comparability*). POSTULATE Ib: $a \,\mathfrak{g}\, b$, $b \,\mathfrak{g}\, c$ *together imply* $a \,\mathfrak{g}\, c$ (*transitivity*). Ia and Ib together state that $\mathfrak{g}$ is a *complete ordering* of A.

We define *tastes* by attaching the values "true," "false," to the statements $a \,\mathfrak{g}\, b$ applied to all pairs of points in A.

The *preference* and *indifference* relations, $\mathfrak{p}$ (read "preferred to" or "better than") and $\mathfrak{i}$ (read "indifferent to" or "equivalent to"), are defined as follows:

$$a \,\mathfrak{p}\, b \text{ means } a \,\mathfrak{g}\, b \text{ and not } b \,\mathfrak{g}\, a,$$
$$a \,\mathfrak{i}\, b \text{ means } a \,\mathfrak{g}\, b \text{ and } b \,\mathfrak{g}\, a,$$

where a, b are real vectors, and $a = b$ means, as usual, $a_\mu = b_\mu$, $\mu = 1, \cdots, \nu$. It follows from the above postulates and definitions that $a = b$ implies $a \,\mathfrak{i}\, b$; that $a \,\mathfrak{i}\, b$ implies $b \,\mathfrak{i}\, a$; and that $\mathfrak{i}$ and $\mathfrak{p}$ are transitive relations.

9. *Indifference sets; utility indices; utility function.* For any a, we define its indifference set $J(a)$ as follows: $J(a)$ consists of all b's such that $b \,\mathfrak{i}\, a$. It follows that every a breaks up the set A into the following three pair-

[6] Such propositions were discussed by Frisch [1, 2], Wold [1], Allen [2], Lange [1, 2], Samuelson [1], Arrow [2], and others.

wise disjunct subsets: $J(a)$; the region of all b's such that $b \mathfrak{p} a$; the region of all c's such that $a \mathfrak{p} c$.

It follows that the indifference sets can be uniquely arranged into a sequence. If a real number $u(a)$ is assigned to the indifference set $J(a)$, one can assign to any indifference set $J(b)$ a real number $u(b)$ such that $u(a) \geqslant u(b)$ means $a \mathfrak{g} b$. It follows that

$$u(a) > u(b) \text{ means } a \mathfrak{p} b,$$

and

$$u(a) = u(b) \text{ means } a \mathfrak{i} b.$$

The numbers $u(a)$, $u(b)$, $\cdots$ are called utility indices or, briefly, utilities; and the function u on set A, a utility function. The class of all utility functions will be denoted by U.

A function $\phi(z)$ is said to be monotonically increasing if $\phi(z_1) > \phi(z_2)$ for $z_1 > z_2$. Let $u(a)$ be a utility function. Then $\phi[u(A)]$ is a utility function if and only if ϕ is a monotonically increasing function.

10. *Maximizing utility.* Define a subset F' of the feasible closed set F (Section 6 above) as follows: if d and b are in F' and if c is in F but not in F' then $d \mathfrak{i} b \mathfrak{p} c$. The set F' consists of the "best feasible" prospects; and since F is closed, F' must contain at least one element (Arrow [1]). We study the case when F' has only one element (the best feasible prospect). Let u be a utility function. Then, under conditions just stated, $u(c) < u(b) = u(d) = \text{Max } u(a)$ over all a in F. To say that a certain function u is a utility function means that $u(a)$ is maximized with respect to a, subject to the condition that a is in the feasible set F.

11. *Internal and external averages.* Let r be a real number, $0 < r < 1$. Then $b = ra + (1 - r)c = \{[ra_0 + (1 - r)c_0], \cdots, [ra_\nu + (1 - r)c_\nu]\}$ will be called an internal average of a and c; it is represented by a point on the straight line segment $\overline{ac}$. Let q be a real number, $q > 1$. Then $d = qa + (1 - q)c$ is called an external average of a and c and lies on the continuation of $\overline{ac}$ beyond a. Obviously, a is an internal average of d and c, with $q = 1/r$.

12. POSTULATE II: *If $a \mathfrak{p} b \mathfrak{p} c$, then there exists r, $0 < r < 1$, such that* $b \mathfrak{i} [ra + (1 - r)c]$. (*Continuity* of the preference relation.)

13. POSTULATE III: *For any a and r, $0 < r < 1$, there exists a prospect b such that* not $a \mathfrak{i} [ra + (1 - r)b]$. That is, the neighborhood of any point a contains at least one point b in A which is not equivalent to a.

14. *Indifference surfaces.* Postulates II and III imply together that the indifference set $J(a)$ forms a boundary between two regions of A: one containing all b's such that $b \mathfrak{p} a$; the other containing all c's such that $a \mathfrak{p} c$. In this case, $J(a)$ has one dimension less than the space, that

is, $\nu - 1$ dimensions, and is called an indifference surface. [Note that in special cases, e.g., for some points a on the boundary of A, the set $J(a)$ may have $< \nu - 1$ dimensions: a may have either no b's ($b \mathfrak{p} a$) or no c's ($a \mathfrak{p} c$) in its neighborhood.] In any case, no two distinct indifference sets $J(a) \neq J(b)$ may intersect, since by the definitions in Section 8, $a \mathfrak{p} b$ is inconsistent with $a \mathfrak{i} b$.

15. Postulate III and its implication that indifference sets are, in general, surfaces forms an interesting link with tradition. But for the present purposes it can be replaced by one of the following two weaker postulates suggested, respectively, by Abraham Wald and Herman Rubin.

POSTULATE III*: *There exist two prospects a and b in the interior of A such that $a \mathfrak{p} b$ or $b \mathfrak{p} a$.*

POSTULATE III′: *There exist at least four distinct indifference sets.*

III. THE THEORY OF ASSETS

16. The theory of assets pointed out that the man chooses between achievable probability distributions. This proposition was just stated in Section 10. However, with exceptions that will be stated below, past writings on the theory of assets[7] did not represent each prospect by a point in the space of probabilities of mutually exclusive "histories" x. They considered instead the joint probability distribution of the $\gamma(\tau + 1)$ quantities of each commodity at each point of time (see Section 3). Each "cell" of such a joint distribution corresponds to some history x. To a given distribution $a(x)$ of the histories x corresponds one joint probability distribution of the alternative commodity-amounts at various points of time. The latter distribution can be characterized by certain parameters, and writers on the theory of assets represented each prospect by a point in the space of these parameters; call this space P. For example, if only one commodity—say, money income—and one point of time are considered, so that the alternative histories x differ only by the amounts of money income, each prospect can be characterized by the mean, the variance, the skewness, etc., of the probability distribution of income; if more than one commodity or time-point is considered, say $\gamma(\tau + 1) = \epsilon$ quantities, each prospect may be characterized by ϵ means, ϵ variances, $\epsilon(\epsilon - 1)/2$ correlations, etc. Each distribution parameter was regarded as being a "good" (or possibly a "bad") in the eyes of the man. Accordingly, each distribution was regarded as a bundle of "commodities," subject to postulates which are

[7] I. Fisher [1], Appendix to Chapter XVI, p. 10. Pigou [1], Appendix 1. Hicks [1, 2]. Makower and Marschak [1]. Marschak [1, 2, 3]. This list is incomplete. See also footnote 10.

analogous to Postulates I–III above. Tastes were described by indifference surfaces drawn in the parameter space P, and the empirical properties of these surfaces were discussed. For example, the risk-aversion (risk-discount), i.e., the rate of substitution between the mean and the variance of income, expressed by the relevant slope of an indifference surface, was stated to be positive. Opportunities were defined as a region of the space P (analogous to our region F of A). The boundary of this region—the "opportunity surface"—was derived from the individual's initial possession of assets (physical goods and positive or negative claims), the prices or other conditions in the market of the assets (perfect or otherwise), and the production conditions. With opportunities and tastes defined in the space of distribution parameters, the old theory of assets defined rational decision analogously to Section 10. (If all functions are well behaved, this will correspond to choosing the point of tangency between the opportunity surface and an indifference surface.) From here, the theory explained the market prices of assets in the same way in which prices of production factors are determined from the indifference maps of individuals, their initial possession of production factors, and the production functions for final goods. It also claimed to explain the determination of optimal balance sheets and plans, in the way the ordinary theory determines the equilibrium flows of consumption.

17. It was pointed out by Tintner[8] that the utility index might be considered, not as a function of distribution parameters but as a functional of the distribution function itself. This is tantamount to the use of the domain of probabilities, A, as in Part I above. To become more than a change in notation, this suggestion will need an additional postulate (Part IV). With the help of such a postulate, it will be shown in Part V that indifference sets in the A-space must be parallel hyperplanes. This will, in turn, imply the possibility of defining utility functions determined up to linear transformations. It will also imply that such a utility index of an uncertain prospect $a \equiv (a_0, \cdots, a_\nu)$ equals its "expected utility," defined as the average of the utility indices of the sure prospects $a^{(0)}, \cdots, a^{(\nu)}$, weighted with the probabilities $a_0, \cdots, a_\nu$. This should also help to revise the theory of determination of optimal plans, defined as rational balance sheets or rational strategies.[9]

18. *The mean vs. other parameters.* It has been common among economists to insist that the means of the amounts of commodities are not the only parameters of their joint distribution that are relevant to the man's decisions, and possibly not the most important ones, and attempts

[8] Tintner [1, 2, 3]; see also Hurwicz [1].

[9] Marschak [7].

were made to specify which additional, or alternative, parameters—e.g., the higher moments[10]—should be considered. These writers, e.g., Marschak [2], Tintner [1], often failed to make clear that the statement "the average *amounts of goods* are not alone relevant to the man's decision" does not contradict the statement that the average *utility* is maximized by him. The latter proposition permits, in fact, to relate "risk-aversion," "advantage of diversification," and similar concepts of the older asset theory to the properties of the utility function of sure prospects.[11]

IV. FOURTH POSTULATE ON CHOICE BETWEEN PROSPECTS

19. POSTULATE IV_1: *If* $a \text{ i } a'$ *and* $0 < r < 1$, *then* $a \text{ i } [ra + (1 - r)a']$.
In words: If prospect a is equivalent to another prospect, a', then the prospect of obtaining either a or a' with certainty is equivalent to each of them. To illustrate this postulate, we can first interpret a, a' as sure prospects (Section 4 above), that is, as two mutually exclusive future sequences of events or, still simpler, as two mutually exclusive bundles of commodities. Suppose a man is indifferent as between a car and a $1,000 bill. Then he should be also indifferent as between a car, a $1,000 bill, and the ticket to a lottery in which the chances of winning a car or a $1,000 bill are, respectively, .2 and .8, say. The same applies if, for a car and $1,000, we substitute securities, or patents, or lottery tickets, etc., promising, respectively (if combined with the man's other assets), prospects such as those defined in Part I.

POSTULATE IV_2: *Let* a, a', b, *be three distinct prospects*; *let* $c = ra + (1 - r)b$, *and* $c' = ra' + (1 - r)b$, *where* $0 < r < 1$; *then* $a' \text{ i } a$ *implies* $c' \text{ i } c$.

In words: If two prospects a and a' are equivalent, then the prospect of having a or b with probabilities r, $1 - r$, and the prospect of having a' or b with the same respective probabilities r, $1 - r$, are equivalent. For example, suppose a $1,000 bill and a car are equivalent. Compare two lottery tickets, one promising a $1,000 bill or a house with probabilities .9 and .1, the other promising a car or a house, also with probabilities

[10] As a rival to the mean, the mode was suggested by Lange [3]. As a measure of riskiness the standard deviation was suggested by I. Fisher [1] as early as 1906. This is in line with the actuarial theory of an individual insurance policy: Cramér [1], Part II. Another important parameter is the probability that a certain variable (cash reserve, annual profit, etc.) fall below a constant, for example, below zero. This occurs in the theory of the total risk of an insurance company: Cramér [1], Part III. According to a verbal communication of H. Somers, this may also be the meaning of a suggestion made by Fellner [1]. Unlike those of Cramér, the suggestions of Lange, Fellner, and other economists often intend to describe actual rather than advisable behavior (see Section 1, above).

[11] See Marshall [1], Menger [1], Friedman and Savage [1], Marschak [4, 7].

.9 and .1. Then there is no reason to prefer one of the two lottery tickets to the other.

For the geometry of these postulates see Figure 2, where a' i a and $\overline{a'a} \parallel \overline{c'c}$. Then by IV_1, the internal average m i a; and by IV_2, c' i c.

Postulate IV: *If* a' i a *and* $0 < r < 1$, *then for any* b, $[ra' + (1 - r)b]$ i $[ra + (1 - r)b]$. This postulate summarizes IV_1 (for $b = a$) and IV_2 (for $b \neq a$).

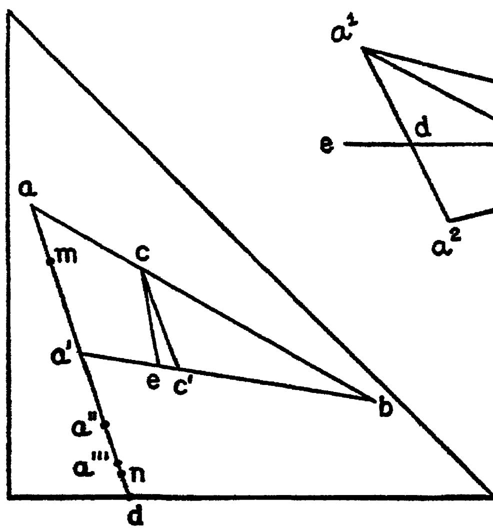

Figure 2

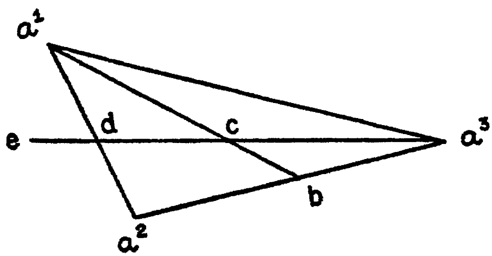

Figure 3

20. While IV_1 states that the internal average (see Section 11 above) of two equivalent prospects is equivalent to them, the following lemma, implied in IV_1 and IV_2, extends this statement to external averages though under a restriction:

Lemma 1: *If* a i a', *and an interior point* p *is collinear with* a, a', *then* p i a.

Proof: Suppose p is between a and a': for example, $p = m$ in Figure 2; then the proposition follows directly from Postulate IV_1. Suppose now a' is between p and a: for example $p = n$ in Figure 2. Then continue the line $\overrightarrow{a'n}$ till its intersection d with the boundary of the domain. Define s, $0 < s < 1$, by the equation $a' = sd + (1 - s)a$. Define $a'' = sd + (1 - s)a'$. Then a i a' implies, by IV_2, a'' i a. Define next $a''' = sd + (1 - s)a''$, $a'''' = sd + (1 - s)a'''$, etc. By applying IV_2 successively, every point a'', a''', a'''', $\cdots$ is shown to be in $J(a)$; and by IV_1 any point between a and a', $\cdots$, a'''', $\cdots$ is also in $J(a)$. Therefore,

since the distances $\overline{da'}, \cdots, \overline{da''''}, \cdots$ approach zero, any point on the segment $\overline{ad}$ is in $J(a)$ except possibly d itself.

V. LINEAR UTILITY FUNCTIONS OF PROSPECTS

21. Part V will develop the implications of our four postulates. A particularly simple system of indifference sets—a family of parallel planes—will be shown to represent tastes with regard to prospects. It will then be proved that there exists a class of linear utility functions of prospects, and that the "expected values" (properly defined) of such functions are maximized over the feasible set of prospects by the rational man (defined as one who follows the rules of logic and arithmetic and the four postulates of Part IV).

THEOREM 1: *The indifference sets $J(a)$ are parallel hyperplanes of $\leqslant \nu - 1$ dimensions. For any interior a, $J(a)$ has $\nu - 1$ dimensions.* This theorem will be derived from Postulates I, II, III*, IV; the results[12] would of course be valid if III* were replaced by a stronger Postulate III. (If, instead of Postulate III*, we admitted the stronger Postulate III and hence the existence of indifference hypersurfaces, Postulate IV would imply immediately that these surfaces must be parallel hyperplanes.)

To give a general idea of the proof we first confine ourselves to $\nu = 2$ and to the interior of A only, and show that then (1) all indifference sets are straight lines and (2) all these lines are parallel. The general case will then be introduced in Section 22:1.

21:1 We first show that if three interior points a^1, a^2, a^3, are equivalent they must be collinear.[13] For suppose that, on the contrary, they are the vertices of a triangle, as in Figure 3. Then, by Lemma 1, any b on the triangle, any c within the triangle, and any interior point e outside the triangle are in $J(a^1)$; and this contradicts III*. Conversely, any interior point on the straight line containing a^1 and a^2 is in $J(a^1)$; this follows from Lemma 1.

21:2. Figure 2 shows that if $J(a)$ and $J(c)$ are straight lines containing the segments $\overline{aa'}$ and $\overline{cc'}$, then $\overline{aa'} \parallel \overline{cc'}$. For suppose that the line drawn through c parallel to $\overline{aa'}$ meets $\overline{ba'}$ in e. Then by IV_2, $e \mathrel{\dot{\imath}} c$; and since $c \mathrel{\dot{\imath}} c'$, the points c, c', e, must be collinear, as shown in 21:1. Hence e must coincide with c'.

22:1. To prove for the whole domain A, of any dimension ν, that each set $J(a)$ is a hyperplane of $\leqslant \nu - 1$ dimensions (of exactly $\nu - 1$ dimensions if a is in the interior of A), Abraham Wald suggested the following set of lemmas. Denote by a' an interior point;[14] denote by $J'(a')$ the set of all interior points in $J(a')$.

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[12] Rubin [1] has used Postulate III′ instead of III or III*. His proof applies to ν finite or infinite.

[13] Superscripts indicate sure prospects only if in parentheses: $a^{(1)}$ not a^1; see Section 5.

[14] We shall denote interior points by single, boundary points by double primes through Sections 22:1–22:4.

22:2. LEMMA 2: *For any interior point a, the set $J'(a')$ consists of all the interior points of a hyperplane, $H(a')$ of $< \nu - 1$ dimensions.* This follows from Lemma 1, as in 21:1, by proving that the vertices of a $(\nu + 1)$–hedron cannot be equivalent, and that any interior point contained in the same hyperplane with ν equivalent points is equivalent to them.

22:3. LEMMA 3: *For any interior point a', $J(a')$ consists of $J'(a')$ and the intersection of $H(a')$ with the boundary of A.* To prove this, consider b'', a point on the boundary of A, and show that b'' is in $J(a')$ if, and only if, b'' is on the hyperplane $H(a')$.

22:3:1. *Neccessity:* Suppose b'' is in $J(a')$ but not in $H(a')$. Then any interior point c' on the segment $\overline{b''a'}$ is not in $H(a')$; but on the other hand, by IV, c' is in $J(a')$ and hence in $J'(a')$ and, by Lemma 2, in $H(a')$. Because of this contradiction, b'' must be in $H(a')$.

22:3:2. *Sufficiency:* To show that if the boundary point b'' is on the hyperplane $H(a')$ it is also on the indifference set $J(a')$, distinguish three cases: (1) there exist two interior points c', d', such that $c' \mathfrak{p} a' \mathfrak{p} d'$; (2) there is no interior point c' such that $c' \mathfrak{p} a'$; (3) there is no interior point d' such that $a' \mathfrak{p} d'$.

In Case (1), by Lemma 2, neither c' nor d' is in $H(a')$. Suppose first (1α): that $b'' \mathfrak{p} a'$; then $b'' \mathfrak{p} a' \mathfrak{p} d'$, and, by Postulate II, there exists a point e' on segment $\overline{b''d'}$ such that e' is in $J(a')$; by Lemma 2, e' is in $H(a')$. But on the other hand, since b'' is in $H(a')$ while d' is not, e' on segment $\overline{b''d'}$ cannot be in $H(a')$. Because of this contradiction, the assumption (1α) that $b'' \mathfrak{p} a'$ is false; so is, by symmetry, the assumption (1β) that $a' \mathfrak{p} b''$. Hence $a' \,\dot{\imath}\, b''$.

In Case (2) there exists, by III*, an interior point d' such that $a' \mathfrak{p} d'$; and, by Lemma 2, d' is not in $H(a')$. Suppose first (2α): that $b'' \mathfrak{p} a'$. Then $b'' \mathfrak{p} a' \mathfrak{p} d'$, and, by II, there exists an interior point e' on segment $\overline{b''d'}$ such that e' is in $J(a')$ and hence, by Lemma 2, in $H(a')$. But on the other hand, since b'' is in $H(a')$ while d' is not, e' is not in $H(a')$. Because of this contradiction, the assumption $b'' \mathfrak{p} a'$ must be false.

Suppose next (2β): that $a' \mathfrak{p} b''$; and consider separately three subcases: $(2\beta\alpha)$: $d' \mathfrak{p} b''$; $(2\beta\beta)$: $b'' \mathfrak{p} d'$; $(2\beta\gamma)$: $b'' \,\dot{\imath}\, d'$. In $(2\beta\alpha)$, $a' \mathfrak{p} d' \mathfrak{p} b''$ and hence, by II, there is a point e' on $\overline{a''b'}$ such that $e' \,\dot{\imath}\, d'$; but then, since both a' and b'' are in $H(a')$, e' is in $H(a')$, while [as stated at the beginning of Case (2)] its equivalent d' is not in $H(A')$. This contradicts Lemma 2; hence $(2\beta\alpha)$ is impossible. Subcase $(2\beta\beta)$ is also impossible: for then $a' \mathfrak{p} b'' \mathfrak{p} d'$ so that, by II, there exists a point e' on $\overline{a'd'}$ such that $e' \,\dot{\imath}\, b''$ and e' is not in $H(a')$. Let $f' = re' + (1 - r)a'$; $g' = rb'' + (1 - r)a'$; $0 < r < 1$. Then, by IV, $f' \,\dot{\imath}\, g'$. But, by Lemma 2, this is impossible since g' is in $H(a')$ (because b'' is), while f' is not in $H(a')$ (because e' is not). There remains subcase $(2\beta\gamma)$: $b'' \,\dot{\imath}\, d'$. This is also impossible since then $rd' + (1 - r)a' \,\dot{\imath}\, rb'' + (1 - r)a'$, $0 < r < 1$; yet the point to the right of $\dot{\imath}$ is, and that to the left is not, in $H(a')$, thus contradicting Lemma 2.

Thus cases (2α): $b'' \mathfrak{p} a'$, and (2β): $a' \mathfrak{p} b''$, are both impossible. There remains (2γ): $b'' \,\dot{\imath}\, a'$.

In Case (3), by a proof similar to that of Case (2), one shows again that $b'' \,\dot{\imath}\, a'$.

22:4 LEMMA 4: *If $b \mathfrak{p} a$, $0 < r < 1$, and $c = ra + (1 - r)b$, then $b \mathfrak{p} c \mathfrak{p} a$.*

PROOF: Since b, c, a, are collinear, it follows from Lemmas 2 and 3 that neither $a \,\dot{\imath}\, c$ nor $b \,\dot{\imath}\, c$ holds. If $a \mathfrak{p} c$ then by II the segment bc has an interior point d such that $a \,\dot{\imath}\, d$. But this is impossible since, by IV, $b \mathfrak{p} a$ would then be false. One shows similarly that $c \mathfrak{p} b$ is impossible. Hence $b \mathfrak{p} c \mathfrak{p} a$.

By Lemmas 2 and 3 the indifference set $J(a')$ containing an interior point a' is a hyperplane, $H(a')$. We shall now prove

22:5. LEMMA 5: *If c is an interior point, $J(c)$ is a hyperplane of dimension $= \nu - 1$.*

PROOF: Suppose the set $J(c)$ [identical with $H(c)$] has $\nu - \alpha > 0$ dimensions, and $\alpha > 0$. Then $H(c)$ is determined by α independent linear equations. If $\alpha > 1$, we can choose a 2-dimensional plane through c that is determined by $\nu - 2$ independent equations such that, when combined with the α independent equations of $H(c)$, they give ν independent equations determining c; that is, K and $H(c)$ intersect in a unique point c. But if $\alpha = 1$, the intersection of $H(c)$ with any 2-dimensional plane is determined by at most $1 + \nu - 2 = \nu - 1$ independent equations, and hence has at least one dimension. We shall now show that $\alpha > 1$ contradicts our previous lemmas. For if c is the only point common to both $H(c)$ and K, then, by Lemma 2, for no point a in K, distinct from c, is $a \;\mathfrak{i}\; c$. Choose three noncollinear points a^1, a^2, a^3, in K such that c is in the interior of the triangle $a^1a^2a^3$ (see Figure 3). Let $a^1 \;\mathfrak{p}\; c$; then $a^2 \;\mathfrak{p}\; c$, $a^3 \;\mathfrak{p}\; c$; for if (for example) $c \;\mathfrak{p}\; a^2$ held, then, by Postulate II, there would be a point d on segment $\overline{a^1a^2}$ such that $d \;\mathfrak{i}\; c$; but this is impossible since the plane K has only one point in common with $H(c)$. Hence either $a^i \;\mathfrak{p}\; c$ $(i = 1, 2, 3)$ or $c \;\mathfrak{p}\; a^i$ $(i = 1, 2, 3)$. The line a^1c intersects the segment $\overline{a^2a^3}$ in b, an internal average of a^2 and a^3. Hence $a^i \;\mathfrak{p}\; c$ $(i = 1, 2, 3)$ implies $b \;\mathfrak{p}\; c$ because both $a^2 \;\mathfrak{p}\; c$ and $a^3 \;\mathfrak{p}\; c$, and because either $a^2 \;\mathfrak{i}\; a^3$ (in which case Lemma 1 applies) or not $a^2 \;\mathfrak{i}\; a^3$ (in which case Lemma 4 applies); but by the same Lemma 4, $a^i \;\mathfrak{p}\; c$ $(i = 1, 2, 3)$ implies $c \;\mathfrak{p}\; b$ because $a^1 \;\mathfrak{p}\; c$. Hence the assumption $a^i \;\mathfrak{p}\; c$ was false. The assumption $c \;\mathfrak{p}\; a^i$ leads (by symmetry) to a similar contradiction; and $a^i \;\mathfrak{i}\; c$ was already ruled out. Hence the assumption $\alpha > 1$ was false, and the lemma is proved.

22:6. LEMMA 6: *If a is on the boundary of A, $J(a)$ is a hyperplane of $\leqslant \nu - 1$ dimensions.* For, if $J(a)$ contains an interior point a', $J(a) = J(a') = H(a')$ has $\nu - 1$ dimensions by Lemma 5. If $J(a)$ contains boundary points only, then $J(a)$ is a hyperplane of $\leqslant \nu - 1$ dimensions (facet, edge, $\cdots$) forming part of the boundary of A.

23. To prove Theorem 1 (Section 2:1 above), it remains to be shown that all the indifference hyperplanes are parallel. This is done by the same reasoning as in 21:2, except that the lines $\overline{aa'}$, $\overline{cc'}$, $\overline{ce}$, have to be regarded as the intersections of the plane of the page with hyperplanes of dimension $\leqslant \nu - 1$.

24. THEOREM 2: *If prospects a^1, a^2, a^3, $\cdots$ lie on a straight line in this order, and if $h > i$ and $j > k$, then $a^h \;\mathfrak{i}\; a^i$ implies $a^j \;\mathfrak{i}\; a^k$, and $a^h \;\mathfrak{p}\; a^i$ implies $a^j \;\mathfrak{p}\; a^k$.*[15] This follows immediately from the above lemmas.

25. Since all indifference sets J are parallel hyperplanes (Figure 4 for $\nu = 2$), we can write an equation for any $J(a)$. Choose any sure prospect, e.g., $a^{(0)}$, as the origin, and let $d[a; a^{(0)}]$ be its distance from the plane $J(a)$ which contains the point a with the coordinates $a_1, \cdots, a_\nu$. Then

$$(25.1) \qquad d[a;\, a^{(0)}] = \sum_1^\nu k_\mu a_\mu, \quad \sum_1^\nu (k_\mu)^2 = 1,$$

where k_μ $(\mu = 1, \cdots, \nu)$ are real numbers (direction cosines) characteristic of $J(a)$ but independent of a and of $a^{(0)}$. In particular, if $a^{(\mu)}$ is

[15] We shall encounter this theorem in Part VIII when discussing the "monotonicity of utility functions of prospects."

another sure prospect, its coordinate $a_\pi^{(\mu)}$ $(\pi = 1, \cdots, \nu) = \delta_{\pi\mu}$, i.e., equals 1 when $\pi = \mu$ and 0 otherwise. Therefore the distance of $J[a^{(\mu)}]$

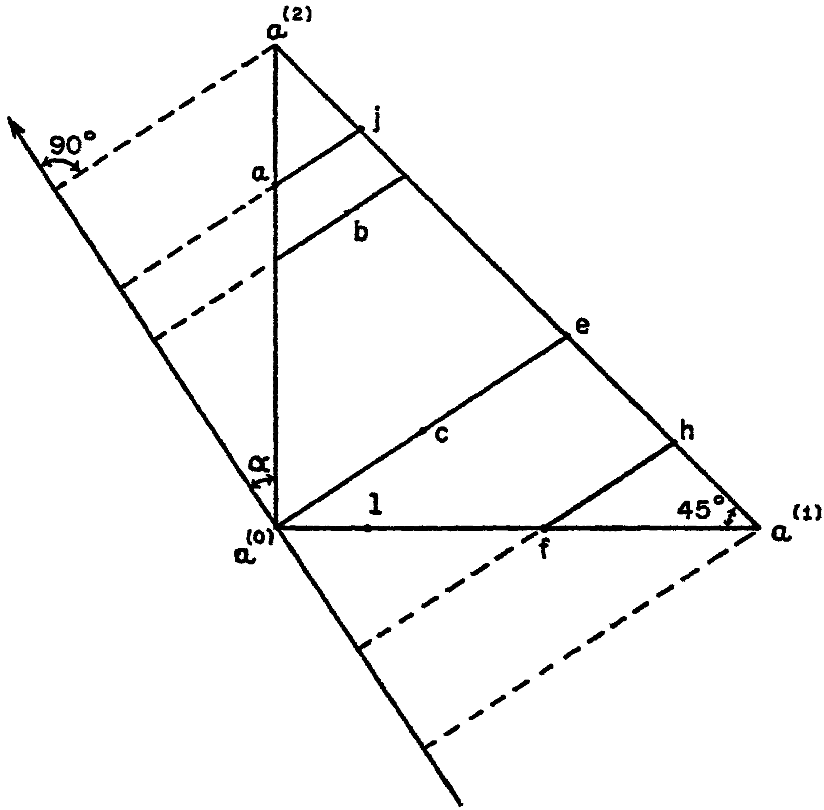

FIGURE 4

from $a^{(0)}$, $d[a^{(\mu)}; a^{(0)}] = \Sigma_{\pi=1}^{\nu} a_\pi^{(\mu)} k_\pi = \Sigma_{\pi=1}^{\nu} \delta_{\pi\mu} k_\pi = k_\mu$. If, in addition, we define $k_0 \equiv 0 = d[a^{(0)}; a^{(0)}]$, we have

$$[da^{(\mu)}; a^{(0)}] = k_\mu, \qquad (\mu = 0, \cdots, \nu). \tag{25.2}$$

If the origin is changed from $a^{(0)}$ to $a^{(\mu)}$, the distance of each indifference hyperplane from the origin is changed by the same constant k_μ. We can therefore replace $d[a; a^{(0)}]$ by the shorter symbol $d(a)$ with the understanding that $d(a)$ is determined up to an additive constant which depends on the arbitrary origin $a^{(0)}$. For further brevity, we shall write

$$d[a^{(\mu)}] \equiv d^{(\mu)}, \qquad (\mu = 1, \cdots, \nu). \tag{25.3}$$

In this notation, and by (25.2), the equation (25.1) of an indifference plane $J(a)$ becomes

$$d(a) = \sum_0^\nu a_\mu d^{(\mu)}. \tag{25.4}$$

26. We shall now show that the distance function $d(a)$ can be used as a utility function, as defined in Section 9 above. For this, we first prove the following lemma.

LEMMA 7: *If* $a \,\mathfrak{p}\, b \,\mathfrak{p}\, c$, *then either* $d(a) > d(b) > d(c)$ *or* $d(a) < d(b) < d(c)$.

PROOF: By Postulate II there exists $b' = ra + (1 - r)c$, $0 < r < 1$, such that b' is in $J(b)$. For each component of b', $b'_\mu = ra_\mu + (1 - r)c_\mu$, $\mu = 0, \cdots, \nu$. Hence, by (25.4),

$$\begin{aligned} d(b) = d(b') = \sum b'_\mu d^{(\mu)} &= r \sum a_\mu d^{(\mu)} + (1 - r) \sum c_\mu d^{(\mu)} \\ &= rd(a) + (1 - r)\, d(c), \end{aligned}$$

thus proving the lemma: the plane $J(b)$ lies between the planes $J(a)$ and $J(c)$.

We can always define the positive direction for distances in such a way that the first and not the second alternative of Lemma 7 holds. We then have for any two prospects a, b:

$$\begin{array}{lll} d(a) > d(b); & d(a) < d(b); & d(a) = d(b); \\ \text{according as } a \,\mathfrak{p}\, b; & b \,\mathfrak{p}\, a; & a \,\mathfrak{i}\, b. \end{array} \tag{26.1}$$

Recalling the definitions in Section 9, we can therefore state as proven the following theorem.

THEOREM 3. *If positive direction is defined by* (26.1), *then the distances* $d(a)$, $d(b)$, $\cdots$ *of the planes* $J(a)$, $J(b)$, $\cdots$ *from the origin* $a^{(0)}$ *form a set of utility indices.*

27. Other statements of Theorem 3 are the following:

The distance function $d(a)$ *is a utility function.*

The man maximizes $d(a)$ *over the feasible set* F.

There exist $\nu + 1$ *numbers* $k_1, \cdots, k_\nu$, *and* $k_0 = 0$, *such that for any* a, b, $\Sigma_0^\nu k_\mu a_\mu >$, $<$, *or* $= \Sigma_0^\nu k_\mu b_\mu$, *according as* $a \,\mathfrak{p}\, b$, $b \,\mathfrak{p}\, a$, *or* $a \,\mathfrak{i}\, b$.

The indifference sets in the space of prospects are parallel planes stacked up in the order of the man's preferences.

28. On the "indifference map" in Figure 4, the arrow indicates positive direction from the origin $a^{(0)}$. As always (compare with Figure 1), $\overline{a^{(0)} a^{(1)}} = \overline{a^{(0)} a^{(2)}} = 1$. Further, in this example

$$a^{(2)} \,\mathfrak{p}\, a \,\mathfrak{p}\, b \,\mathfrak{p}\, a^{(0)} \,\mathfrak{i}\, c \,\mathfrak{p}\, f \,\mathfrak{p}\, a^{(1)};$$

$$\begin{aligned} &0 < \alpha < \pi/2;\ d^{(2)} = \cos\alpha > d(a) = \overline{aa^{(0)}} \cos\alpha > d(b) \\ &\quad > d^{(0)} = 0 = d(c) = d(e) > d(f) = -\overline{fa^{(0)}} \sin\alpha \\ &\quad = d(h) > d^{(1)} = -\sin\alpha, \text{ etc.} \end{aligned} \tag{28.1}$$

Note that the scale unit is determined by the fact that for any a, $\Sigma_0^\nu a_\mu = 1$, while the origin is fixed by choosing $d^{(0)} = 0$.

29. Theorems 1 to 3 are derived from Postulates I, II, III*, and IV on *rational* behavior. They cannot be tested by observation. But we can sketch out the following statements on *actual* behavior which, in principle, can be so tested:

(1) "Faced with a varying set of feasible prospects, Mr. Smith makes decisions which, with a probability p, do not deviate by more than a preassigned distance from the decisions consistent with Theorem 3." For example, with $\nu = 2$, let $a^{(0)}$, $a^{(1)}$, $a^{(2)}$ be the prospects of a sure life income of, respectively, \$2,000, \$5,000, and \$20,000. The varying sets of feasible prospects can consist of various pairs or triplets, etc., of points marked in Figure 4. A set of decisions of Mr. Smith is inconsistent with the four postulates whenever it is inconsistent with relations (28.1), after fixing $d'^{(0)} = 0$. A set of decisions indicating that there is no angle α such that $d^{(2)} = \cos\alpha$, $d(f) = -\overline{fa^{(0)}} \sin\alpha = d(h)$, would indicate deviations from rationality as defined by the postulates. The procedure of testing whether Smith is "tolerably rational," i.e., whether p exceeds a certain limit, is a statistical one; so is the procedure of estimating the parameters of the indifference map (in our example, the angle α). The difficulties are obvious and stem from the economist's inability to make experiments. If Smith is made to answer questions on his decisions when faced with hypothetical sets of feasible prospects, the answers may involve a bias (and not merely a random deviation) when compared with his actual behavior. If, on the other hand, the sets of feasible prospects are those that have actually faced him in his life, they may not yield themselves to convenient simplification; moreover, Smith may have changed his tastes in the course of his life, although they had remained consistent during each of several shorter periods. Some of these difficulties are obviated, but replaced by others, if the following hypothesis is tested:[16]

(2) "At least a fraction q of the buyers and sellers of real estate, securities, insurance policies, and other assets are tolerably rational (in the sense defined above); and the numbers characterizing their indifference maps (viz., their respective angles α, in the example of Figure 4) have such and such a frequency distribution."

30. The μth sure prospect can be denoted both as $a^{(\mu)}$ and as $x^{(\mu)}$; its utility function will be denoted by

$$u^{(\mu)} \equiv u[a^{(\mu)}] \equiv u[x^{(\mu)}]. \tag{30.1}$$

[16] See in particular Friedman and Savage [1], Section 6; Vickrey [1]; and Törnqvist [1].

$x^{(\mu)}$ is a particular value of the random vector x, and $u^{(\mu)}$ is a particular value of the random scalar $u(x)$. The expected value of $u(x)$ given the distribution $a(x)$ can be written[17] as

$$\text{(30.2)} \qquad \mathcal{E}u(x) \mid a(x) \equiv \sum_{\mu=0}^{\nu} u[x^{(\mu)}]\, a_\mu \equiv \sum_{\mu=0}^{\nu} u^{(\mu)}\, a_\mu \equiv \mathcal{E}u \mid a;$$

in the last form no dummy variables (x, μ) appear, thus making clear that the quantity in question depends only on the utility function u and the distribution ("prospect") a. The quantity $\mathcal{E}u \mid a$ can be regarded as the result of an operation which is performed upon the function u and depends on a; or as the result of an operation which is performed upon the prospect a and depends on u. According to the emphasis needed we may call $\mathcal{E}u \mid a$ the "expected value of the utility function u," as well as the "expected utility for a."

31. In this notation and by Theorem 3, the distance function (25.4) has the following properties: $d(a)$ is a utility function; and $\mathcal{E}d \mid a = d(a)$. Consequently, $\mathcal{E}d \mid a$ is a utility function. That is, the man maximizes the expected value of utility function d. In Section 9 above, we have defined U, the class of utility functions. Now define a subclass V of U, consisting of all functions v with the following properties:

$$(\alpha_1) \qquad \mathcal{E}v \mid a \text{ is in } U,$$

$$(\alpha_2) \qquad \mathcal{E}v \mid a \text{ is in } V,$$

and

$$(\alpha_3) \qquad \mathcal{E}v \mid a = v(a).$$

Note that property (α_3) would suffice to define V, since (α_3) implies (α_2), and (α_2) implies (α_1). We have just shown that V contains at least one element, viz., the distance function d: it satisfies (α_3). We shall now prove the following theorem.

32. Theorem 4: *V consists of all linear increasing functions of d.* That is, a function v is in V if and only if there exist two numbers l, m $(m > 0)$ such that for any a

$$(\beta) \qquad v(a) = l + md(a).$$

Proof: *Sufficiency*: If (β) is true, then by (25.4) and the definitions (30.2),

$$\mathcal{E}v \mid a = \sum v^{(\mu)} a_\mu = l + m \sum d^{(\mu)} a_\mu = l + m\, d(a) = v(a).$$

[17] Henceforth all summations over μ will be from 0 through ν.

Thus (β) implies all three α-properties in Section 31.

Necessity: Suppose (α_1), the weakest of those three properties, is true. Then since both $\mathcal{E}v \mid a$ and $d(a)$ are in U, there exists a monotone increasing function ϕ, such that

$$(32.1) \qquad \mathcal{E}v \quad a = \phi[d(a)] \equiv \phi\, d(a)$$

(omitting brackets after ϕ for brevity). Put $a = a^{(\mu)}$. Then, since $\mathcal{E}v \mid a^{(\mu)} = v^{(\mu)}$, we have, by (32.1), $v^{(\mu)} = \phi\ d^{(\mu)}$; $\phi\ d(a) = \mathcal{E}v \mid a = \Sigma\, a_\mu v^{(\mu)} = \Sigma\, a_\mu \phi\, d^{(\mu)}$; thus

$$(32.2) \qquad \begin{cases} \sum a_\mu \phi\, d^{(\mu)} - 1\cdot\phi\, d(a) = 0; & \\ \sum a_\mu\, d^{(\mu)} - 1\cdot d(a) = 0 & \text{[by (25.4)];} \\ \sum a_\mu \cdot 1 - 1\cdot 1 = 0 & \text{[by (4.1)].} \end{cases}$$

The three equations (32.2) form a homogeneous linear system identical in $a_0, \cdots, a_\nu, -1$. Hence the rows

$$\begin{array}{ccc} \phi\, d^{(0)}, \cdots, & \phi\, d^{(\nu)}, & \phi\, d(a) \\ d^{(0)}, \cdots, & d^{(\nu)}, & d(a) \\ 1, \cdots, & 1, & 1 \end{array}$$

are linearly dependent. Therefore ϕ is a linear function,

$$\phi\, d(a) = l + md(a);$$

and $m > 0$ since ϕ was already stated to be an increasing function. Thus (α_1) implies (β); and since (β) was shown (in the sufficiency proof) to imply (α_2) and (α_3), the proof is complete.

33. Thus V consists of all increasing linear functions of the probabilities $a_0, \cdots, a_\nu$:

$$(33.1) \qquad v(a) = l + m \sum a_\mu\, d^{(\mu)}.$$

We shall call the function $v(a)$ a *linear utility function* and the scalar $v(a)$ a *linear utility index* of prospect a. If Figure 4 represents an indifference map, and the utility index is measured perpendicularly to the plane of the page, then any plane whose intersection with the plane of the page is parallel would represent some element of V, or its negative.

The simplest linear utility function is the distance function $d(A)$ itself. In this case $l = 0$, $m = 1$. The distance function is a linear utility

function unique up to an additive constant that depends on the choice of the origin such as $a^{(0)}$.

34. Immediately from (33.1) we obtain the well-known property of linear functions summarized in

THEOREM 5: *Let prospect a be a linear combination of ρ prospects, $b^1, \cdots, b^\rho$: $a = \Sigma_1^\rho r_\pi b^\pi$. If v is a linear utility function, then and only then*

$$v(a) = \sum_1^\rho r_\pi v(b^\pi). \tag{34.1}$$

In particular, if a is an average, $a = r_1 b^1 + r_2 b^2$, $r_1 + r_2 = 1$,

$$v(a) = r_1 v(b^1) + r_2 v(b^2) \tag{34.2}$$

and

$$\frac{v(b^1) - v(a)}{v(a) - v(b^2)} = \frac{r_2}{r_1}. \tag{34.3}$$

Moreover, since $a = \Sigma_0^\nu a_\mu a^{(\mu)}$, we obtain again the property (α_3) in (31):

$$v(a) = \sum_0^\nu a_\mu v^{(\mu)} = \mathcal{E}v \mid a \equiv \mathcal{E}v(x) \mid a(x). \tag{34.4}$$

35. On the other hand, the following property is not exclusive to linear utility functions: *If v_1 and v_2 are two linear utility functions, then there exist two numbers l, m ($m > 0$) such that for any prospect a,*

$$v_2(a) = l + mv_1(a). \tag{35.1}$$

That is, v_1 and v_2 are (increasing) linear transforms of each other; or, in other formulations, the function v is unique up to a linear transformation or up to two linear coefficients. This property follows from (33.1) but is obviously possessed also by any class W consisting of utility functions w such that for any a

$$w(a) = l + m\phi_w d(a), \tag{35.2}$$

where $m > 0$, and ϕ_w is a monotone increasing function, *not necessarily linear*; for example, ϕ_w may be the cubic function.

36. Another property which follows from (33.1) yet is possessed by

certain classes of nonlinear utility functions is this: *If v_1 and v_2 are two linear utility functions, then*

$$(36.1) \qquad v_2(a) - v_2(b) = m[v_1(a) - v_1(b)],$$

where m is a positive constant independent of a, b. That is, the difference between the linear utilities of two prospects is unique up to a proportionality factor. This property follows from the property (35.1), and is therefore also shared by a class of nonlinear utility functions such as W defined in (35.2).

VI. MEASURABILITY AND MANAGEABILITY

37. The question has been often asked whether utility is "measurable in the sense in which temperature is."[18] Because temperature readings are nothing but distances (or angles or volumes depending on the design of the thermometer), and in order to avoid discussing the relation between the absolute and other scales of temperature, I prefer comparison with altitudes. The altitude of a point on the earth's surface is unique up to two constants (one positive) depending on the origin, such as the sea level, and the unit, such as the meter. The linear utility index of a prospect is also unique up to two constants (one positive), l and m in (33.1). In this sense, altitudes expressed in meters, and linear utility indices, are equally strictly measurable.

38. This property they share, however, with altitudes expressed in cubes of their meter measurements, and with utility indices obtained from linear ones by substituting, say, $[d(a)]^3$ for $d(a)$; see Section 35. The property of being unique up to two constants is weaker than certain other properties of linear utility functions, for example, the property (34.4) of expected values of linear utility indices.

Numbers of meters, and not cubes of those numbers, are used in measuring altitudes because this simplifies certain important operations. One derives the altitude of point Y in meters from the altitude of point Z in meters and from the number of meters contained in the upward horizontal projection of the line connecting the two points, by forming an algebraic sum of the latter two numbers. If cubes of those numbers were used instead, one would have to cube the sum of two cube roots, a more complicated operation.

Money amounts are unique up to one positive proportionality factor depending on the currency unit. Their logarithms are also determined up to a single constant, an additive one. For additive operations, dollars are more manageable than logarithms. But the latter may be preferable when, as in constructing graphs of hyperinflation, one is interested in

[18] Allen [2], Lange [1], von Neumann and Morgenstern [1].

ratios; just as (I understand) it is convenient to use logarithms of time in discussing radioactive disintegration.

39. Thus, linear utility indices of prospects share with altitudes expressed in units of length not only the particular kind of measurability (viz., up to two constants, one positive) but also the "manageability" with respect to certain, viz., linear, operations. In the case of utility indices of prospects, a particularly important kind of linear operations is the forming of averages as in (34.2), and especially the forming of expected values as in (34.4). For no other but linear utility indices is it true that the utility of a distribution (a prospect) is equal to the expected value of utilities of sure prospects, computed on the basis of that distribution.[19]

40. In Part V implications were derived from our four postulates on rational behavior. These results can be stated in two equivalent forms:

(α) The indifference surfaces in prospect space are linear and parallel (Theorem 1).

(β) There exist linear utility functions of prospects, with certain convenient properties (Theorems 3–5).

We have just shown that (α), (β)—and therefore our four postulates—are sufficient to derive measurable and manageable (in the defined sense) utility functions. To see whether these postulates are also necessary, we shall weaken (α) and study the implications. Suppose then that the indifference sets were not parallel planes [cf. (25.1)],

$$\sum_0^{\nu} k_\mu a_\mu = d(a), \qquad k_0 = 0, \tag{40.1}$$

but would satisfy, instead, a more general equation

$$\sum_0^{\nu} k_\mu \psi(a_\mu) = d_\psi(a), \qquad k_0 = 0, \qquad k_\mu \geqslant 0, \quad (\mu = 1, \cdots, \nu), \tag{40.2}$$

where $d_\psi(a)$ is the utility index (a "generalized distance") of $J(a)$, and ψ is a monotone increasing function such that $\psi(0) = 0$, $\psi(1) = 1$. [For example, if $\psi(z) = z^2$, then indifference surfaces are concentric ellipsoids.] Equation (40.2) is consistent with our postulates, except Postulate IV [unless $\psi(z) = z$]. Statements analogous to those in Section 25 and Section 26 would hold. Putting $a = a^{(\mu)}$, (40.2) becomes

$$k_\mu = d_\psi[a^{(\mu)}] \equiv d_\psi^{(\mu)}, \tag{40.3}$$

say; substituting in (40.2), we find that the set $J(a)$ is defined by the utility index

[19] Compare Arrow [2], Chapter II, Section 1.

$$d_\psi(a) = \sum_0^\nu \psi(a_\mu) d_\psi^{(\mu)}. \tag{40.4}$$

Define the operator $\mathfrak{S}_\psi$ by

$$\mathfrak{S}_\psi u \mid a \equiv \mathfrak{S}_\psi u(x) \mid a(x) \equiv \sum_0^\nu \psi(a_\mu) u[a^{(\mu)}]; \tag{40.5}$$

this is a generalization of the operator $\mathfrak{S}$, each probability a_μ being replaced by its transform $\psi(a_\mu)$. Then, by (40.4),

$$d_\psi(a) = \mathfrak{S}_\psi d_\psi \mid a. \tag{40.6}$$

Now, in analogy to the class V in Section 31, define a class V_ψ consisting of all utility functions v_ψ such that, for any a, $v_\psi(a) = \mathfrak{S}_\psi v_\psi \mid a$. Then V_ψ consists of all products of the "generalized distance" with arbitrary (and positive) constants, m. For, if $v_\psi = md_\psi$, the definition (40.5), with v_ψ substituted for u, implies $\mathfrak{S}_\psi v_\psi \mid a = v_\psi(a)$. Conversely, if v_ψ is in V_ψ, and if ϕ is an unknown monotone function such that $\phi d(a) \equiv \phi[d(a)] = \mathfrak{S}_\psi v_\psi a$, we obtain a system analogous to (32.2) but with only two instead of three equations (since the transforms of probabilities need not add up to unity):

$$\begin{aligned} \sum_\mu \psi(a_\mu)\phi\, d_\psi^{(\mu)} &= \phi\, d_\psi(a), \\ \sum_\mu \psi(a_\mu)\, d_\psi^{(\mu)} &= d_\psi(a). \end{aligned} \tag{40.7}$$

We see that if Postulate IV were replaced in such a way as to lead to (40.2) instead of (40.1), it would be possible to define a utility function $v_\psi(a)$ unique up to a proportionality factor. Thus $v_\psi(a)$ is even more strictly "measurable" than the function $v(a)$ derived with the aid of Postulate IV. In this sense, the virtue of that postulate does not consist in the "measurability" of the function $v(a)$. Rather, it consists in the "manageability" of $v(a)$, in the sense of property (34.4). In the case of the utility function $v(a)$ the expected value $\mathfrak{S}v \mid a \equiv \Sigma v^{(\mu)} a_\mu$ is maximized; while in the case of the function $v_\psi(a)$, not the expected value, but a more complicated, usually nonlinear, expression $\mathfrak{S}_\psi v_\psi \mid a \equiv \Sigma_\mu v_\psi^{(\mu)} \psi(a_\mu)$ is maximized.

41. This opens the question (to be taken up again in Part VIII) of still further, or of alternative, generalizations of (40.1) and, consequently, of the appropriate generalization of underlying postulates. Each such generalization would possibly define a nonlinear analogon—say $\mathfrak{S}_*$—of the operation $\mathfrak{S}$ and an analogon—say V_*—of the linear subclass V of utility functions. V_* may be empty or consist of one element only,

or of two or more functions which are, possibly nonlinear, transforms ϕ_* of each other. The possible nonlinearity of the operations $\mathcal{E}_*$ and ϕ_* that replace the operations $\mathcal{E}$ and ϕ of Section 30 will render the utility functions of prospects less "manageable." One might call them less "measurable" if ϕ_* involved more parameters than ϕ; but in Section 40 we just had a case where the opposite was true.

VII. COMPARISON WITH VON NEUMANN AND MORGENSTERN

42. The word prospect is occasionally used in the *Theory of Games and Economic Behavior* (e.g., on top of page 18) essentially in the same sense as in this paper (see also Hicks [2]). More often the authors speak of a "combination of events." We call a prospect the probability distribution of the time sequences of all combinations of commodities (or, in fact, of any kinds of events), while von Neumann and Morgenstern take examples involving two or three commodities at one time point only. Since our time sequences are by necessity mutually exclusive, no confusion can arise in our formulations as to the effect of complementarity upon the choices. If proper wording is used, this is not an essential difference.

43. *Verbal presentation: utility differences.* In the verbal presentation (Section 3.3) of the *Theory of Games*, the authors use as a behavior postulate the equivalent of our (34.3). In the present paper, on the other hand, (34.3) was derived from four postulates on rational behavior. Von Neumann and Morgenstern (in their verbal presentation) do not have Postulate IV but seem to have what amounts to Postulates I and II.

Our decision to introduce Postulate IV, rather than to start with (34.3) as a behavior postulate, is dictated by the desire to avoid behavior postulates which are neither immediately plausible nor show themselves as approximated by easily observable action. Postulates I and III* state essentially that man chooses between prospects; and we see men actually making choices—determining the balance sheet by buying or selling, or by abstaining from buying and selling, assets. Postulate II excludes discontinuities. Postulate IV is also a very weak (i.e., very plausible) one; it merely rules out behavior of a kind which most people would call absurd. The statement (34.3), on the other hand, is neither immediately plausible nor is it amenable to easy observation. True, a person when interviewed may make comparisons between two utility differences, but the comparison does not show itself in any choice except in the choice to answer a certain question in a certain way. However, H. Chernoff pointed out that such a check of utility comparison by an actual choice could be made in the following situation. According to L. J. Savage, the rational strategy in a one-person game under complete ignorance of

circumstances might be defined as one that results in minimaxing "regret," i.e., the difference between the utility actually realized and the utility that would have been realized if all circumstances were known. (See Marschak [6a] where this postulate was used.) That is, the strategy must be chosen so that even under circumstances which yield for this strategy a regret not smaller than under any other circumstances, this regret is not larger than for any other strategy.[20] Thus, a rational man playing the one-person game under ignorance of future circumstances has to base his choices on comparing differences between utilities.

44. *Mathematical presentation: the chain of implications.* In their mathematical presentation, the authors of the *Theory of Games* do not actually make use of a behavior postulate on utility differences.[21] The behavior postulates used are set up in their Section 3.6, and we shall refer to them presently. An extensive appendix published in the second edition of the book shows that these postulates result in propositions (3:1:a) and (3:1:b) of the text (Section 3.5). The second of these implies, and is implied by, the linearity of what we called the utility functions of prospects, and appears thus equivalent to our Theorems 1–5. From this, the "measurability" of utility is derived in (3:5:b) and (3:6) of the *Theory of Games*, in the sense that utility is unique up to a linear transformation: a property weaker than the preceding one, as shown in this paper (Sections 35–41 above).

44.1. *The axioms.* The difference between the presentation in this paper and in the *Theory of Games* thus boils down to the difference between our Postulates I–IV, and the group of axioms (3:A), (3:B), (3:C), of the *Theory of Games.*[22] The "entities" u, v, of the *Theory of*

[20] Contrast this with the two-person zero-sum game. There the rational player obtains the minimax utility, not the minimax difference between two utilities. (Von Neumann and Morgenstern [1], Sections 3–15.)

[21] For sure prospects, R. Frisch [1] derived utility increments measurable up to a proportionality factor, by introducing postulates on the individual's choice between small displacements occurring in two different points of the commodity space.

[22] We reprint these axioms from Section 3.6.1 (p. 26) of the *Theory of Games*: "*3.6.1* . . . consider a system U of entities $u, v, w, \cdots$. In U a *relation* is given, $u > v$, and for any number α, $(0 < \alpha < 1)$, an *operation*

$$\alpha u + (1 - \alpha)v = w.$$

These concepts satisfy the following axioms:

(3:A) *$u > v$ is a complete ordering of U.*

This means: Write $u < v$ when $v > u$. Then:

(3:A:a) For any two u, v one and only one of the three following relations holds:

$$u = v, \quad u > v, \quad u < v.$$

Games correspond to the various indifference sets—say J, J', in the present paper. The symbol $>$ defined in the authors' (3:A) is related to our $\mathfrak{p}$ as follows: if J contains a and J' contains a', and J, J' are distinct, we say $a\,\mathfrak{p}\,a'$ or $a'\,\mathfrak{p}\,a$; Von Neumann and Morgenstern say (replacing J, J' by u, v) $u > v$ or $v > u$. Moreover, $u = v$ corresponds to $J = J'$. As to the symbol $+$, and the operation $\alpha u + (1 - \alpha)\, v = w$, the authors define it implicitly by the axioms (except that α is a real scalar, $0 < \alpha < 1$, identical with our r); while we use the symbol $+$ and the operation $ra + (1 - r)a'$ in their usual meaning (addition, multiplication) as applied to the real vectors a, a' and the scalar r. However, select element a from J, and a' from J'. Then the following operation on indifference sets can be defined: $rJ + (1 - r)\, J'$ means $J[ra + (1 + r)a']$. With this interpretation Axioms (3:A) and (3:B) of the *Theory of Games* will follow (with u, v, α replacing J, J', r) if the indifference sets are parallel hyperplanes, i.e., if our Postulates I, II, III*, and IV are fulfilled. In particular, the axiom (3:C:b) can be rendered as follows: "Let $0 < r < 1$, $0 < q < 1$; let a, b, c, e be prospects such that $b = ra + (1 - r)e$, $c = qb + (1 - q)e$. Then $c\ \mathfrak{i}\ [qra + (1 - qr)e]$." This "double mixing" is indeed implied in Postulate IV, as a succession of two single mixings discussed there.

I hesitate to say which (if any) is the stronger system of axioms. I find it easier to translate Postulate IV immediately into the language of observable decisions—see the examples in Section 19 above—than to do so with von Neumann and Morgenstern's (3:C:b), whether stated as it is in terms of entities not susceptible to ordinary arithmetical operations, or restated in the form I have just suggested. To use a

(3:A:b) $u > v$, $v > w$ imply $u > w$.
(3:B) *Ordering and combining.*
(3:B:a) $u < v$ implies that $u < \alpha u + (1 - \alpha)v$.
(3:B:b) $u > v$ implies that $u > \alpha u + (1 - \alpha)v$.
(3:B:c) $u < w < v$ implies the existence of an α with

$$\alpha u + (1 - \alpha)v < w.$$

(3:B:d) $u > w > v$ implies the existence of an α with

$$\alpha u + (1 - \alpha)v > w.$$

(3:C) *Algebra of combining.*

(3:C:a) $$\alpha u + (1 - \alpha)v = (1 - \alpha)v + \alpha u.$$

(3:C:b) $$\alpha[\beta u + (1 - \beta)v] + (1 - \alpha)v = \gamma u + (1 - \gamma)v$$

where $\gamma = \alpha\beta$."

On what follows, I had the privilege of consulting Abraham Wald.

phrase of von Neumann and Morgenstern, the approach used here seems to be more "transparent," and its postulates have a more "immediate intuitive meaning by which [their] appropriateness may be judged directly."

44.2. *The proofs.* As already mentioned in Section 44, a part of the route, the part after establishing the proposition on the linearity of (certain) utility functions, is common to both presentations. But the previous part of the route, between the axioms and that proposition, is different. The present author does not claim to improve upon the minutiousness and rigor of the appendix to the second edition of the *Theory of Games.* The present paper consists essentially in applying the technique of indifference curves—as stated in our Postulates I–III—to prospects which are not sure. When Postulate IV is added, this technique yields in a simple way the Neumannian results. These are also obtained if III is replaced by a weaker Postulate III* (or III′).

VIII. LOVE OF DANGER INCOMPATIBLE WITH THE FOUR POSTULATES

45. THEOREM 6: *If* $a \mathfrak{p} b$ *and* $c = ra + (1 - r)b$, *then* $a \mathfrak{p} c$ *for* $r > 1$, $c \mathfrak{p} a$ *for* $r > 1$. (That is, a is preferred to every point between a and b and to every point on the prolongation of the segment $\overline{ab}$ beyond b; and every point on the prolongation of that segment beyond a is preferred to a.) This property of prospects follows directly from Theorems 2 or 4. In Figure 4, take any three points on a straight line not parallel to the indifference lines (planes)! The property may be called the monotonicity of the utility function of prospects $u(a)$, on the space A. It is different from the monotonicity assumption sometimes made for the utility function of commodities—say, $w(x)$ on the space X. This assumption is stated formally as follows (cf. Arrow [1]):

> It is always possible to define final goods $x_1, \cdots, x_\epsilon$ in such a way that if $x^{(\alpha)} \equiv (x_1^{(\alpha)}, \cdots, x_\epsilon^{(\alpha)})$ and $x^{(\beta)} \equiv (x_1^{(\beta)}, \cdots, x_\epsilon^{(\beta)})$ and if $x_\gamma^{(\alpha)} \geqslant x_\gamma^{(\beta)}$ for all γ and $x_\delta^{(\alpha)} > x_\delta^{(\beta)}$ for some δ $(\gamma, \delta = 1, \cdots, \epsilon)$, then $x^{(\alpha)} \mathfrak{p}\, x^{(\beta)}$.

That is, the addition of any final "good" to (or subtraction of any final "bad" from) a bundle of commodities (a history, a sure prospect) increases the utility of the bundle. More precisely, this assumption is taken to be valid not over the whole space X but only over a so-called effective region of it;[23] outside of this region the man is saturated with one or more of the "goods," so that "goods" become "bad" or at least indifferent.

Obviously this form of monotonicity cannot apply to the prospect space A, because of restriction (4.1): it is impossible to increase the

[23] Allen [1], H. Wold [1].

probability of one of the histories, say, the probability a_μ of $x^{(\mu)}$, without decreasing the probability of at least one of the other histories. Also, it would be impossible to classify each history—possibly a whole course of life—as either "good" or "bad" in the way we classify single final commodities such as bread and leisure or labor and sickness. The monotonicity assumption for commodities says: "The more of a good thing the better. If opportunities are unlimited, increase the quantity of every good thing until it ceases to be a good thing." The monotonicity assumption for prospects says: "If one alternative is better than another, increase the probability of the former at the expense of the latter.[24] If opportunities are unlimited, choose the prospect that promises the best history with 100% probability": the corner $a^{(2)}$ in Figure 4.

The monotonicity assumption for commodities is usually stated as an independent postulate or as the definition of the effective region. The monotonicity assumption for prospects (Theorem 6) follows from Postulates I, II, III (or III* or III′), and IV.

46. If our four postulates define rational behavior, the following actual behavior is not rational: many men, not at all bent on suicide, are enthusiastic mountain climbers and are elated, not (or not only) by exercise and scenery but by the very danger, in the following sense. Suppose the probability of fatal accident is 5%. The climber may prefer a survival chance of 95% to one of, say, 80% *but also* to one of 100%! The utility index of prospects reaches in this case its maximum at a point which is not a corner of domain A. (If in Figure 4, $a^{(0)}$ and $a^{(2)}$ are two ways of continuing to live, and $a^{(1)}$ is death, the highest utility will be not at $a^{(0)}$ or $a^{(2)}$ or a but, say, at j or l or b.)

47. Such behavior contradicts the monotonicity Theorem 6, and therefore at least one of our postulates. It is, therefore, in general inconsistent with the existence of linear utility functions for prospects. It also follows that the lover of danger does not, in general, maximize the expected value of the utility index. Not for him the linear utility functions of prospects, and the measurable and "manageable" (Part VI) utility indices. Love of danger in this sense may very well be present also in what is usually considered economic decisions. The danger of loss including ruin, though probably shunned in the conservative code or cant of business, has quite possibly added to the zest and desirability of many an historically important venture, in the career of the leaders of mercenary armies, in the financing of great geographic discoveries or, closer to our time, in the financing of inventions and theater plays, and in stock and commodity speculation.

[24] This verbal proposition is easiest to visualize in the special case when the alternatives considered are sure prospects. Theorem 6 is more general since it holds for all prospects.

48. The "love of danger" defined as nonmonotonicity of the utility function for prospects is to be carefully distinguished from the phenomenon alluded to in Part III: whether high variance of random income (or of the random quantity of any commodity) is or is not desirable depends on whether the marginal utility of mean income (or commodity) happens to be increasing rather than decreasing. Neither a positive nor a negative sign of the "risk-discount" contradicts the monotonicity of the utility function for prospects (nor, of course, the monotonicity assumption for goods).[25]

The statement "a man may desire to gamble" has thus two meanings. If it means that the sign of the risk-discount is indefinite, the statement is consistent with rational behavior as defined by the four postulates. (In fact, it is well explained by implications from these very postulates.) If it means, however, that the man "loves danger" so that his utility function of prospects is not monotonic along a straight line in space A, drawn at an angle to the indifference planes, then the statement is not consistent with the four postulates. Should one revise the postulates of rational choice in order to be able to include love of danger under rational behavior, then it would probably become impossible to attain manageable utility indices in the sense of Part VI. As indicated in Section 41, the operation of maximizing expected value would have to be replaced by a more general or more unwieldy one. The alternative is to stamp love of danger as nonrational, on a par with, say, the behavior of a man who prefers a to b and b to c and yet prefers c to a (denial of transitivity Postulate I); or a man to whom a car and a \$1,000 bill are equivalent and who yet prefers one of these things to a lottery ticket promising with certainty that either the one or the other thing will be obtained (denial of Postulate IV_1).

49. At this point we can only hint at what is probably the most important virtue of the advice to maximize expected utility and, hence, of the behavior postulates implied in this advice. We conjecture that, for a large class of distribution functions and utility functions, the following proposition is true: if every action is chosen in such a way as to maximize the expected utility, then, as the number of such actions is increased, the probability that the achieved utility differs from the maximum achievable utility by an arbitrarily small number, approaches unity. If this proposition is true, and if the logic of rational behavior is used as a tool of advice (see Section 1 above), it is best to exclude love of danger as irrational. If, on the other hand, rational behavior is used as an approximation hypothesis for the study of actual behavior, then again, love of danger may be considered as a departure from rationality. It is then a matter of empirical psychology and is important to economics in the same way in which inconsistency (intransitivity) of preferences, inability

[25] To show this, expand $\mathcal{E}u(x)$ about the mean of x, where x = income.

to measure, to imagine, to remember, are important. We treat it in the same way that we treat the fact that people make mistakes of arithmetic and infringe upon the rules of conventional logic.

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INVERSION OF THE LEONTIEF MATRIX BY POWER SERIES[1]

By Frederick V. Waugh

The Leontief matrices of inter-industry transactions are large, a row for each industry in a nation. It would be desirable to invert such matrices of an order of from 100 to 200. The present paper suggests using the sum of a power series to approximate the inverse of a Leontief matrix with any desired degree of accuracy. This requires many more multiplications than do such direct methods as the Gauss–Doolittle process. But the method proposed in this paper is especially well adapted to automatic computation on the new electronic machines, in which case the large number of multiplications is not serious. The main advantage of the proposed method is that it provides an upper bound to the error of any element in the estimated inverse. A short cut method is also indicated for computing the approximation when the number of terms of the power series needed to obtain the desired degree of accuracy is large.

INTRODUCTION

World War II confronted economists with the enormous job of testing the feasibility of proposed production goals, and of determining the adjustments needed in various industries to reach these goals. This job is less spectacular in peace times, but it is important nonetheless. Most nations are developing programs to assure full employment. In the United States, the Employment Act of 1946 directs the President to report (among other things) "the levels of employment, production, and purchasing power obtaining in the United States and such levels needed to carry out the policy"; that is, "to promote maximum employment, production, and purchasing power." To do this job it is necessary to explore in some detail the alternative patterns of potential output of the United States, to find which patterns are consistent with national objectives, and to show what adjustments would be needed in industry and in agriculture to reach the desired pattern of output.

So far, at least, the most promising method of attacking problems of this kind appears to be that developed by Leontief [4–8] of Harvard and used in a series of studies by Cornfield, Evans, and Hoffenberg [2, 3], of the Bureau of Labor Statistics.

As explained in detail below, these studies are based upon a set of inter-industry equations, of which the matrix of coefficients has come to be called the "Leontief matrix." There are two general problems

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concerning such matrices: first, that of determining the coefficients of the structural equations; and second, that of solving the set of equations in order to determine the output in each industry required to reach one or more sets of objectives for final domestic consumption and for exports. The papers of Leontief, Cornfield, Evans, and Hoffenberg, already cited, discuss in some detail the problem of estimating the coefficients. The present paper is concerned with the second problem, that of obtaining a general solution for the system of equations.

Basic Notation and Assumptions

Let the economy consist of n industries and the $(n + 1)$–st ("residual") sector (final domestic consumption and exports). We use the following notation:

x_i = output of the ith industry, $(i = 1, 2, \cdots, n)$,
x_{ij} = the part of x_i sold to the jth industry, $(i, j = 1, 2, \cdots, n)$,
y_i = the part of x_i sold to the "residual" sector,
p_i = the price per unit of the product of the ith industry.

By definition, we have

$$(1.1) \qquad x_i \geqslant 0, \qquad x_{ij} \geqslant 0, \qquad y_i \geqslant 0, \qquad p_i \geqslant 0,$$

that is, quantities and prices are nonnegative.

It is assumed that

$$(1.2) \qquad x_i > 0, \qquad (i = 1, 2, \cdots, n),$$

that is, no industry is completely idle.

All outputs x_i are *net*, so that

$$(1.3) \qquad x_{ii} = 0, \qquad (i = 1, 2, \cdots, n),$$

that is, on a net basis an industry buys nothing from itself.

The units of measurement may be either "conventional" (tons, bushels, etc.) or in "dollar's worth." In the latter case we have

$$(1.4) \qquad p_i = 1, \qquad (i = 1, 2, \cdots, n),$$

that is, when we measure in "dollar's worth" units the price is always one dollar per unit. For the former case see Appendix.

The following matrix notation will be used. Column vectors (n-rowed):

$$(1.5) \qquad x = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \cdot \\ \cdot \end{pmatrix}, \quad e = \begin{pmatrix} 1 \end{pmatrix}$$

Diagonal matrices (square, n–rowed):

$$(1.6)\quad D_x = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & x_n \end{pmatrix}, \quad P = D_p = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & p_n \end{pmatrix}.$$

Other matrices (square, n–rowed):

$$(1.7)\qquad X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \cdot & \cdot & \cdot \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$

the latter to be defined below. A matrix identity, to be used below:

$$(1.8)\qquad D_x e = x.$$

The Technical Coefficient Matrix A

We define the technical coefficient matrix A by the matrix relation

$$(2.1)\qquad A = XD_x^{-1}$$

where D_x is nonsingular by virtue of (1.2). (In nonmatrix notation we have $a_{ij} = x_{ij}/x_j$.)

Two important properties of A may be noted here:

$$(2.2)\qquad a_{ii} = 0, \qquad (i = 1, 2, \cdots, n)'$$

since $x_{ii} = 0$ [by (1.3)]; and

$$(2.3)\qquad a_{ij} \geqslant 0,$$

since it is a ratio of nonnegative numbers [by (1.1)].

The Problem

Since all (net) output goes to one of the other industries or to the residual sector we have the identity

$$(3.1)\qquad x = Xe + y.$$

Using (2.1) we obtain

$$(3.2)\qquad x = AD_x e + y,$$

which by virtue of the identity (1.8) becomes

$$(3.3)\qquad x = Ax + y$$

or

$$(3.4)\qquad (I - A)x = y$$

where, it will be remembered, A is "hollow" [see (2.2)].

The central problem of this paper is to show how (3.4) can be solved for x, given A and y [or in nonmatrix terminology,[2] how (3.5′) can be solved for $x_1, x_2, \cdots, x_n$ given the coefficients a_{ij} and the $y_1, \cdots, y_n$]. It is, therefore, the problem of determining the output for each industry given the objectives (or forecasts) for the "residual" sector (final domestic consumption and exports), assuming that the technology (i.e., the matrix A of coefficients a_{ij}) is known.[3]

The Contribution of This Paper

The problem of solving (3.4) is essentially one of matrix inversion, since we can write the solution in the form

$$x = (I - A)^{-1}y. \tag{4.1}$$

We are thus confronted with the problem of inverting the *Leontief matrix* $I - A$. This is a very large matrix.[4] It would be desirable to work with an industrial classification of 100 to 200 groups, hence with matrices of from 100 to 200 rows and columns. If such large matrices are inverted

[2] In nonmatrix form we would have corresponding relations as follows:

$$x_i = \sum_{j=1}^{n} x_{ij} + y_i, \qquad (i = 1, 2, \cdots, n). \tag{3.1′}$$

Using (2.1) we obtain

$$x_i = \sum_{j=1}^{n} a_{ij}x_j + y_i, \qquad (i = 1, 2, \cdots, n), \tag{3.2′}$$

which can be written as

$$\sum_{j=1}^{n} (\delta_{ij} - a_{ij})x_j = y_i, \qquad (i = 1, 2, \cdots, n), \tag{3.4′}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

and

$$a_{ii} = 0, \qquad (i = 1, 2, \cdots, n),$$

because of (2.2). We can therefore rewrite (3.4′) without using the summation sign as

$$\begin{array}{c} x_1 - a_{12}x_2 - \cdots - a_{1n}x_n = y_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -a_{n1}x_1 - a_{n2}x_2 - \cdots + \quad x_n = y_n \end{array} \tag{3.5′}$$

[3] In practice, a stronger assumption is made, namely that A (i.e., the coefficients a_{ij}) is known and *constant*. In principle only the knowledge of A is required, but if A were variable (a function of x) we would rarely know A without at the same time knowing x. (Statistical estimates of A based on statistical records, theoretical analyses, and knowledge of current or prospective changes in technology are available, of course, not the "true" values.)

[4] Under the assumptions stated below, the matrix $I - A$ is nonsingular.

by direct methods, such as the Gauss–Doolittle process, we run the risk of large errors due to the compounding of smaller rounding errors. Moreover, it is difficult to set any useful upper bound to such errors.

In what follows a method is given for approximating, with any preassigned degree of accuracy, the matrix $(I - A)^{-1}$. This method approximates $(I - A)^{-1}$ by the sum of the first m terms of the power series

$$I + A + A^2 + \cdots + A^{m-1} + \cdots \tag{4.2}$$

and is based on the fact that under the assumptions made (see especially the next section), we have

$$(I - A)^{-1} = \lim_{m \to \infty} E_m , \tag{4.3}$$

where

$$E_m \equiv \sum_{p=0}^{m-1} A^p. \tag{4.4}$$

The bounds of the approximation error are also given.

There are real advantages to such a method when an electronic computer, or any machine capable of multiplying matrices automatically, is available. One could put A into the machine together with instructions, start the machine, and get back the final answer with assurance that errors are less than any preassigned amount.

The first few terms of the power series (4.2) have been used in some studies of the Bureau of Labor Statistics to give an approximate solution for x. In practice, this has been done by computing successively $Iy = y$, $(I + A)y$, $(I + A + A^2)y$, $(I + A + A^2 + A^3)y$, and so on. In the cases actually tried this has, after several steps, reached apparently stable approximations of x. But the power series (4.2) does not converge for all matrices, and so far as we know its convergence has not heretofore been established for the Leontief matrix, nor have upper bounds previously been set to the error involved in stopping at any particular point.

A PROPERTY TO BE USED IN CONVERGENCE PROOF

If the "dollar's worth" units are used in measuring the x_i, x_{ij}, and y_j [so that $p_i = 1$, cf. (1.4)], we find that

$$v_j \equiv x_j - \sum^{n} x_{ij} \tag{5.1}$$

is the familar "value added"[5] in the jth industry.

Assuming that each industry has a positive value added, so that

[5] v_j consists of payrolls, profits, taxes, etc.

$v_j > 0$, $(j = 1, 2, \cdots, n)$, we use the fact that[6] $x_{ij} = a_{ij}x_j$ [see (2.1)] and obtain from the above the inequality

$$x_j - \sum_{i=1}^{n} x_{ij} \equiv \left(1 - \sum_{i=1}^{n} a_{ij}\right) x_j > 0. \tag{5.2}$$

Hence, because $x_j > 0$ [see (1.2)] and because the a_{ij} are nonnegative [see (2.3)], we obtain from (5.2) the inequality

$$\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} |a_{ij}| < 1 \tag{5.3}$$

which will be of fundamental importance in the subsequent proof of convergence.[7]

PROOF OF CONVERGENCE

To demonstrate the convergence of (4.2) and to derive error limits we make use of the norm of A, or $N(A)$. For this purpose we shall use Bowker's [1] definition

$$N(A) = \max_j \sum_i |a_{ij}|; \tag{6.1}$$

that is, the norm of any matrix may be defined as the largest sum of the absolute values of elements in any column of the matrix. In the case of this particular matrix, we have seen in (2.3) that no element is negative, so the norm could be written simply

$$N(A) = \max_j \sum a_{ij}. \tag{6.2}$$

One important property of this norm is apparent from its definition (6.1): no element of the matrix can be larger than the norm of the matrix. That is,

$$a_{ij} \leqslant N(A). \tag{6.3}$$

Two other properties of the norm that are essential here are

$$N(A + B) \leqslant N(A) + N(B) \tag{6.4}$$

and

$$N(AB) \leqslant N(A) \cdot N(B). \tag{6.5}$$

Note that (6.5) implies

$$N(A^k) \leqslant [N(A)]^k. \tag{6.6}$$

[6] Note that because of the units used the a_{ij} are pure numbers.

[7] However, (5.3) above is not a necessary condition for the convergence in (4.2).

Properties (6.3), (6.4), and (6.5) hold for any definition of the norm. Bowker proved that (6.1) is a norm and hence that these properties hold for the expression defined by (6.1).

In the case of the Leontief matrix, $I - A$, (5.3) may now be written

$$N(A) < 1. \tag{6.7}$$

The proof of convergence follows easily. Thus we may write

$$\begin{aligned} N(I + A + A^2 + A^3 + \cdots) & \\ &\leqslant N(I) + N(A) + N(A^2) + N(A^3) + \cdots \\ &\leqslant 1 + N(A) + [N(A)]^2 + [N(A)]^3 + \cdots \qquad (6.8) \\ &\leqslant \frac{1}{1 - N(A)}, \end{aligned}$$

since $N(I) = 1$ and $N(A)$ is a scalar less than 1. We have, accordingly, an upper limit to the norm of the sum of the power series in (4.2). Also, it may be noted that since $N(A^{k+1}) \leqslant N(A^k)$ when $N(A) \leqslant 1$, the norm of each successive term in the power series is smaller than the norm of the preceding term; hence as the series is extended to high powers of A, the norm of an additional term becomes closer to zero—thus each element of A^k approaches zero as k increases.

ERROR CONTROL

We propose to use (4.4) as an approximation of the inverse of $I - A$. Here we investigate the error in such an approximation.

The error can be of two kinds: errors of computation, and errors involved in neglecting the powers of A higher than $A^{(m-1)}$. Errors of computation can be kept at a minimum in two ways. First, in multiplying any two matrices $(A^r)(A^s)$ it is good practice to carry as a check row the sums of elements in columns of A^r. This check row is multiplied by columns of A^s, giving a row vector. The equality of elements of this row vector to the sum of elements of the corresponding columns of $A^{(r+s)}$ provides a check of the accuracy of the multiplication. Second, a further, and more detailed, check may be had by obtaining a power of A in two different ways, thus discovering whether there is a discrepancy in any element. The first check would not catch an error in some element of $A^{(r+s)}$ if it should be exactly offset by another error in the same column. But powers of A beyond the second can be computed in at least two ways. Thus, $A^3 = AA^2 = A^2A$, $A^4 = A^2A^2 = AA^3$, etc. By the use of check rows and by computing powers of A in two different ways it should be possible to reduce errors of computation to negligible amounts.

With respect to the second type of error, the error matrix D_m is by definition the difference between (4.2) and (4.4), or

$$D_m \equiv (I - A)^{-1} - E_m = A^m + A^{m-1} + A^{m-2} + \cdots \tag{7.1}$$
$$= A^m(I + A + A^2 + \cdots).$$

Using (6.5), (6.8), (6.6), and the fact that $N(I) = 1$, we find that the norm of D_m,

$$N(D_m) \leqslant \frac{N(A^m)}{1 - N(A)} \leqslant \frac{[N(A)]^m}{1 - N(A)}. \tag{7.2}$$

Inequalities (7.2) give two possible upper bounds to the error norm. Mr. Herman Chernoff, one of the referees of this paper, suggested a third bound that is lower than either of those in (7.2). He pointed out that (7.1) can be written

$$D_m = A^m(I + A + \cdots + A^{m-1}) + A^{2m}(I + A + \cdots + A^{m-1}) + \cdots$$
$$= E_m(A^m + A^{2m}$$

Thus,

$$N(D_m) \leqslant \frac{N(E_m) \cdot N(A^m)}{1 - N(A^m)} \leqslant \frac{N(E_m) \cdot N(A) \cdot N(A^{m-1})}{1 - N(A) \cdot N(A^{m-1})}, \tag{7.3}$$

where all the terms to the right are available after obtaining E_m.

After E_m has been computed, the error norm can be computed either from the middle term of (7.2) or from the last term of (7.3). In general, (7.3) will give a finer estimate.

But it is also possible to use the last term of (7.2) to compute an upper bound to the error *before* computing any terms of the power series. In fact, we can set any chosen bound, say ϵ, and determine the number of terms of the power series that assures

$$N(D_m) \leqslant \frac{[N(A)]^m}{1 - N(A)} \leqslant \epsilon. \tag{7.4}$$

This inequality holds if

$$m \geqslant \frac{\log \epsilon + \log [1 - N(A)]}{\log N(A)}. \tag{7.5}$$

The value of m in (7.5) can be computed before work starts. Then it should be possible for the electronic computer to do all the numerical work automatically, providing an approximate inverse E_m which includes no element with an error of more than ϵ.

If convergence is slow, or if great precision is needed, we can reduce the number of necessary steps of computation by using the equation

$$(I + A)(I + A^2)(I + A^4) \cdots (I + A^{2^r}) = I + A + A^2 + A^3 + \cdots + A^{(2^{r+1}-1)}. \tag{7.6}$$

For example, if (7.5) should indicate that 25 terms of the power series were needed in a particular case to assure that the error were within the needed limit, ϵ, it would not be necessary to compute all the terms $A^2, A^3, \cdots, A^{24}$. Instead we could compute only A^2, A^4, A^8, and A^{16}. Then by (7.6) we obtain

$$(I + A)(I + A^2)(I + A^4)(I + A^8)(I + A^{16}) = I + A + A^2 + A^3 + \cdots + A^{31},$$

or the sum of the power series taken five terms beyond the necessary limit.

ERROR LIMITS FOR APPROXIMATED VECTOR, x

Sometimes we may wish to set an upper bound to the error of approximating (or forecasting) the vector x in (3.4), rather than in the elements of the inverse of $I - A$. The solution of (3.4) for any given y vector is (4.1). But $(I - A)^{-1} = E_m + D_m$, and D_m is unknown. Our approximation of x is $E_m y$ and the error of approximation is $D_m y$.

We have given three different upper bounds for the norm of D_m [inequalities (7.2) and (7.3)]. From (6.5) we have

$$N(D_m y) \leqslant N(D_m) \cdot N(y). \tag{8.1}$$

Thus if the sum of the elements of y is 150,000,000 and if $N(D_m) \leqslant 0.00001$, as measured by (7.2) or by (7.3), we know that the largest possible error in the approximations of output is 1,500.

NONNEGATIVE CHARACTERISTICS OF THE POWER SERIES

Since each element of A is nonnegative, and since the elements of A^2 are

$$a_{ij}^{(2)} = \sum_{r=1}^{n} a_{ir} a_{rj},$$

it follows that each element of A^2 is nonnegative. Moreover, if s is any positive integer, and if each element of A^s is nonnegative, each element of A^{s+1} is nonnegative, since

$$a_{ij}^{(s+1)} = \sum_{r=1}^{n} a_{ir}^{(s)} a_{rj}.$$

So, by induction, all positive integral powers of A are composed entirely of nonnegative elements.

Since $(I - A)^{-1}$ is the sum of the identity matrix, I, and the sum of the infinite series of positive integral powers of A, each element of $(I - A)^{-1}$ is nonnegative. [Hence, also, all negative integral powers of $(I - A)$ are composed of nonnegative elements.]

The nonnegative character of elements of $(I - A)^{-1}$ shows that, given positive values of the y's in (3.4), we will get positive approximations of x. The fact that all positive integral powers of A are composed entirely of nonnegative elements shows that each element of E_m in (4.4) approaches the true value of the corresponding element of the matrix monotonically as m increases. In short, E_m never overestimates the value of any element of the inverse. The amount of underestimate can, however, be reduced to any desired level by including enough terms in the power series.

A SIMPLE NUMERICAL EXAMPLE

The nature of the computations required by the proposed method can be illustrated by applying it to a very simple numerical case.

Let

$$A = \begin{pmatrix} 0.0 & 0.8 \\ 0.6 & 0.0 \end{pmatrix}$$

so that

$$I - A \equiv \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix} - \begin{pmatrix} 0.0 & 0.8 \\ 0.6 & 0.0 \end{pmatrix} = \begin{pmatrix} 1.0 & -0.8 \\ -0.6 & 1.0 \end{pmatrix}.$$

We then have

$$(I - A)^{-1} = \frac{1}{0.52} \begin{pmatrix} 1.0 & 0.8 \\ 0.6 & 1.0 \end{pmatrix} = \begin{pmatrix} 1.923077 & 1.538462 \\ 1.153846 & 1.923077 \end{pmatrix}.$$

In such a simple case direct inversion is obviously quicker and easier than the computation of several terms of the power series. But a consideration of the power series in this simple case may help to clarify the nature of the proposed process.

Shown below are the first four powers of A in this case, together with the corresponding estimates of the inverse, E_{k+1}, and the corresponding error matrices, D_{k+1}, obtained by subtracting E_{k-1} from the known inverse.

k	A^k	E_{k+1}	D_{k+1}
1	$\begin{pmatrix} 0.0 & 0.8 \\ 0.6 & 0.0 \end{pmatrix}$	$\begin{pmatrix} 1.0 & 0.8 \\ 0.6 & 1.0 \end{pmatrix}$	$\begin{pmatrix} 0.923077 & 0.738462 \\ 0.553846 & 0.923077 \end{pmatrix}$
2	$\begin{pmatrix} 0.48 & 0.00 \\ 0.00 & 0.48 \end{pmatrix}$	$\begin{pmatrix} 1.48 & 0.80 \\ 0.60 & 1.48 \end{pmatrix}$	$\begin{pmatrix} 0.443077 & 0.738462 \\ 0.553846 & 0.443077 \end{pmatrix}$

k	A^k	E_{k+1}	D_{k+1}
3	$\begin{pmatrix} 0.000 & 0.384 \\ 0.288 & 0.000 \end{pmatrix}$	$\begin{pmatrix} 1.480 & 1.184 \\ 0.888 & 1.480 \end{pmatrix}$	$\begin{pmatrix} 0.443077 & 0.354462 \\ 0.265846 & 0.443077 \end{pmatrix}$
4	$\begin{pmatrix} 0.2304 & 0.0000 \\ 0.0000 & 0.2304 \end{pmatrix}$	$\begin{pmatrix} 1.7104 & 1.1840 \\ 0.8880 & 1.7104 \end{pmatrix}$	$\begin{pmatrix} 0.212677 & 0.354462 \\ 0.265846 & 0.212677 \end{pmatrix}$

Evidently the approximation E_{k+1} is slowly converging toward the true inverse, but additional terms of the power series would be needed to provide accuracy even to one decimal place. If we should use as an approximation of the inverse, $E_5 = I + A + A^2 + A^3 + A^4$, the norm of the error matrix would be $N(D_5) = 0.354462 + 0.212677 = 0.567139$. Let us see whether this is within the bounds set by (7.2) and (7.3). To do this we use $N(A) = 0.8$, $N(A^4) = 0.2304$, and $N(E_5) = 2.8944$. Moreover, $N(A^5) \leqslant N(A)\cdot N(A^4) = 0.18432$.

Using these values in (7.2) we get $0.567139 \leqslant 0.9216 \leqslant 1.6384$ and by (7.3) we get $0.567139 \leqslant 2.8944\ (0.18432)/0.81568 = 0.654050$. It will be seen that the norm of the error matrix is within the bounds set by (7.2) and (7.3). Inequality (7.3) gives a much more useful bound than (7.2).

To get a more useful approximation of $(I - A)^{-1}$ we might use (7.5) with $\epsilon = 0.005$, assuming that we need to be sure that no element of E_m is in error by more than this amount. According to (7.5) we will be sure of this much precision, at least, if

$$m \geqslant \frac{-2.30103 - 0.69897}{-.09691} < 31,$$

so we can be sure of the required accuracy if E_{31} is properly computed. This would take too long if computed by (4.2), but we find by (7.6) that

$$E_{32} = (I + A)(I + A^2)(I + A^4)(I + A^8)(I + A^{16}).$$

The terms, A, A^2, and A^4 are already available since they were used earlier. A^8 and A^{16} are, respectively,

$$\begin{pmatrix} 0.053084 & 0.000000 \\ 0.000000 & 0.053084 \end{pmatrix} \text{ and } \begin{pmatrix} 0.002818 & 0.000000 \\ 0.000000 & 0.002818 \end{pmatrix}.$$

Thus by (7.6) we have

$$E_{32} = \begin{pmatrix} 1.0 & 0.8 \\ 0.6 & 1.0 \end{pmatrix}\begin{pmatrix} 1.48 & 0.00 \\ 0.00 & 1.48 \end{pmatrix}\begin{pmatrix} 1.2304 & 0.0000 \\ 0.0000 & 1.2304 \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1.053084 & 0.000000 \\ 0.000000 & 1.053084 \end{pmatrix}\begin{pmatrix} 1.002818 & 0.000000 \\ 0.000000 & 1.002818 \end{pmatrix}$$

$$= \begin{pmatrix} 1.923062 & 1.538450 \\ 1.153837 & 1.923062 \end{pmatrix}.$$

Comparing this to the known inverse we find the error matrix,

$$D_{32} = \begin{pmatrix} 0.000015 & 0.000012 \\ 0.000000 & 0.000015 \end{pmatrix}.$$

The norm of the error matrix, $N(D_{32})$ is then 0.000027, which meets the proposed standard of accuracy.

APPENDIX ON UNITS OF MEASUREMENT

In some limited studies it may be possible and desirable to measure outputs in such physical terms as numbers of houses, tons of steel, and bushels of wheat. But when attempting to cover the entire economy of a nation such measures of physical output become impracticable, simply because of the enormous number of commodities and services that are produced. It is necessary to group commodities, and the most convenient grouping is by industries. The most convenient measure of output is the dollar value of gross sales. This measure has two important advantages. First, data in dollar terms for hundreds of industries are available in Census reports, in financial statements of corporations, and in special studies. Second, dollar value is a universal measure, applying to output in any industry, and is additive as between industries. From the computational viewpoint also, the important inequality (5.3) holds when the "dollar's worth" units are used. For these reasons most of the statistical work on Leontief matrices to date has been based upon dollar value of output, rather than on physical units. For similar reasons the y vector, measuring amounts sold to final consumers and to exports, has usually been measured in terms of dollar values.

If, however, some other ("conventional") units have been used for some of the industries (so that not all $p_i = 1$), it is useful to have formulae for going from the "conventional" (tons, bushels, etc.) to the "dollar's worth" units.

We have $x = Px^*$, $D_x = PD_x^*$, $X = PX^*$, $y = Py^*$, where asterisks are attached to quantities measured in "conventional" units while the corresponding quantities measured in "dollar's worth" units are without asterisks. Therefore, if the problem is given to us in "conventional" units (i.e., in terms of y^* and A^*), we must solve for x^* in $(I - A^*)x^* = y^*$ where

$$A^* = X^*(D_x^*)^{-1} = (P^{-1}X)(P^{-1}D_x)^{-1} = P^{-1}(XD_x^{-1})P = P^{-1}AP.$$

To do this we compute $A = PA^*P^{-1}$ and invert $I - A$ by methods described in the text. We then obtain

$$\begin{aligned} x^* &= (I - A^*)^{-1}y^* = [P^{-1}(I - A)P]^{-1}y^* \\ &= P^{-1}(I - A)^{-1}Py^*. \end{aligned}$$

Council of Economic Advisers

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THE BARGAINING PROBLEM[1]

By John F. Nash, Jr.

A new treatment is presented of a classical economic problem, one which occurs in many forms, as bargaining, bilateral monopoly, etc. It may also be regarded as a nonzero-sum two-person game. In this treatment a few general assumptions are made concerning the behavior of a single individual and of a group of two individuals in certain economic environments. From these, the solution (in the sense of this paper) of the classical problem may be obtained. In the terms of game theory, values are found for the game.

INTRODUCTION

A two-person bargaining situation involves two individuals who have the opportunity to collaborate for mutual benefit in more than one way. In the simpler case, which is the one considered in this paper, no action taken by one of the individuals without the consent of the other can affect the well-being of the other one.

The economic situations of monopoly versus monopsony, of state trading between two nations, and of negotiation between employer and labor union may be regarded as bargaining problems. It is the purpose of this paper to give a theoretical discussion of this problem and to obtain a definite "solution"—making, of course, certain idealizations in order to do so. A "solution" here means a determination of the amount of satisfaction each individual should expect to get from the situation, or, rather, a determination of how much it should be worth to each of these individuals to have this opportunity to bargain.

This is the classical problem of exchange and, more specifically, of bilateral monopoly as treated by Cournot, Bowley, Tintner, Fellner, and others. A different approach is suggested by von Neumann and Morgenstern in *Theory of Games and Economic Behavior*[2] which permits the identification of this typical exchange situation with a nonzero sum two-person game.

In general terms, we idealize the bargaining problem by assuming that the two individuals are highly rational, that each can accurately compare his desires for various things, that they are equal in bargaining skill, and that each has full knowledge of the tastes and preferences of the other.

[1] The author wishes to acknowledge the assistance of Professors von Neumann and Morgenstern who read the original form of the paper and gave helpful advice as to the presentation.

[2] John von Neumann and Oskar Morgenstern, *Theory of Games and Economic Behavior*, Princeton: Princeton University Press, 1944 (Second Edition, 1947), pp. 15-31.

natural, therefore, to use utility functions for the two individuals which assign the number zero to this anticipation. This still leaves each individual's utility function determined only up to multiplication by a positive real number. Henceforth any utility functions used shall be understood to be so chosen.

We may produce a graphical representation of the situation facing the two by choosing utility functions for them and plotting the utilities of all available anticipations in a plane graph.

It is necessary to introduce assumptions about the nature of the set of points thus obtained. We wish to assume that this set of points is compact and convex, in the mathematical senses. It should be convex since an anticipation which will graph into any point on a straight line segment between two points of the set can always be obtained by the appropriate probability combination of two anticipations which graph into the two points. The condition of compactness implies, for one thing, that the set of points must be bounded, that is, that they can all be inclosed in a sufficiently large square in the plane. It also implies that any continuous function of the utilities assumes a maximum value for the set at some point of the set.

We shall regard two anticipations which have the same utility for any utility function corresponding to either individual as equivalent so that the graph becomes a complete representation of the essential features of the situation. Of course, the graph is only determined up to changes of scale since the utility functions are not completely determined.

Now since our solution should consist of *rational* expectations of gain by the two bargainers, these expectations should be realizable by an appropriate agreement between the two. Hence, there should be an available anticipation which gives each the amount of satisfaction he should expect to get. It is reasonable to assume that the two, being rational, would simply agree to that anticipation, or to an equivalent one. Hence, we may think of one point in the set of the graph as representing the solution, and also representing all anticipations that the two might agree upon as fair bargains. We shall develop the theory by giving conditions which should hold for the relationship between this solution point and the set, and from these deduce a simple condition determining the solution point. We shall consider only those cases in which there is a possibility that both individuals could gain from the situation. (This does not exclude cases where, in the end, only one individual could have benefited because the "fair bargain" might consist of an agreement to use a probability method to decide who is to gain in the end. Any probability combination of available anticipations is an available anticipation.)

Let u_1 and u_2 be utility functions for the two individuals. Let $c(S)$ represent the solution point in a set S which is compact and convex and includes the origin. We assume:

6. If α is a point in S such that there exists another point β in S with the property $u_1(\beta) > u_1(\alpha)$ and $u_2(\beta) > u_2(\alpha)$, then $\alpha \neq c(S)$.

7. If the set T contains the set S and $c(T)$ is in S, then $c(T) = c(S)$.

We say that a set S is symmetric if there exist utility operators u_1 and u_2 such that when (a, b) is contained in S, (b, a) is also contained in S; that is, such that the graph becomes symmetrical with respect to the line $u_1 = u_2$.

8. If S is symmetric and u_1 and u_2 display this, then $c(S)$ is a point of the form (a, a), that is, a point on the line $u_1 = u_2$.

The first assumption above expresses the idea that each individual wishes to maximize the utility to himself of the ultimate bargain. The third expresses equality of bargaining skill. The second is more complicated. The following interpretation may help to show the naturalness of this assumption: If two rational individuals would agree that $c(T)$ would be a fair bargain if T were the set of possible bargains, then they should be willing to make an agreement, of lesser restrictiveness, not to attempt to arrive at any bargains represented by points outside of the set S if S contained $c(T)$. If S were contained in T this would reduce their situation to one with S as the set of possibilities. Hence $c(S)$ should equal $c(T)$.

We now show that these conditions require that the solution be the point of the set in the first quadrant where $u_1 u_2$ is maximized. We know some such point exists from the compactness. Convexity makes it unique.

Let us now choose the utility functions so that the above-mentioned point is transformed into the point (1, 1). Since this involves the multiplication of the utilities by constants, (1, 1) will now be the point of maximum $u_1 u_2$. For no points of the set will $u_1 + u_2 > 2$, now, since if there were a point of the set with $u_1 + u_2 > 2$ at some point on the line segment between (1, 1) and that point, there would be a value of $u_1 u_2$ greater than one (see Figure 1).

We may now construct a square in the region $u_1 + u_2 \leqslant 2$ which is symmetrical in the line $u_1 = u_2$, which has one side on the line $u_1 + u_2 = 2$, and which completely encloses the set of alternatives. Considering the square region formed as the set of alternatives, instead of the older set, it is clear that (1, 1) is the only point satisfying assumptions (6) and (8). Now using assumption (7) we may conclude that (1, 1) must also be the solution point when our original (transformed) set is the set of alternatives. This establishes the assertion.

We shall now give a few examples of the application of this theory.

EXAMPLES

Let us suppose that two intelligent individuals, Bill and Jack, are in a position where they may barter goods but have no money with which

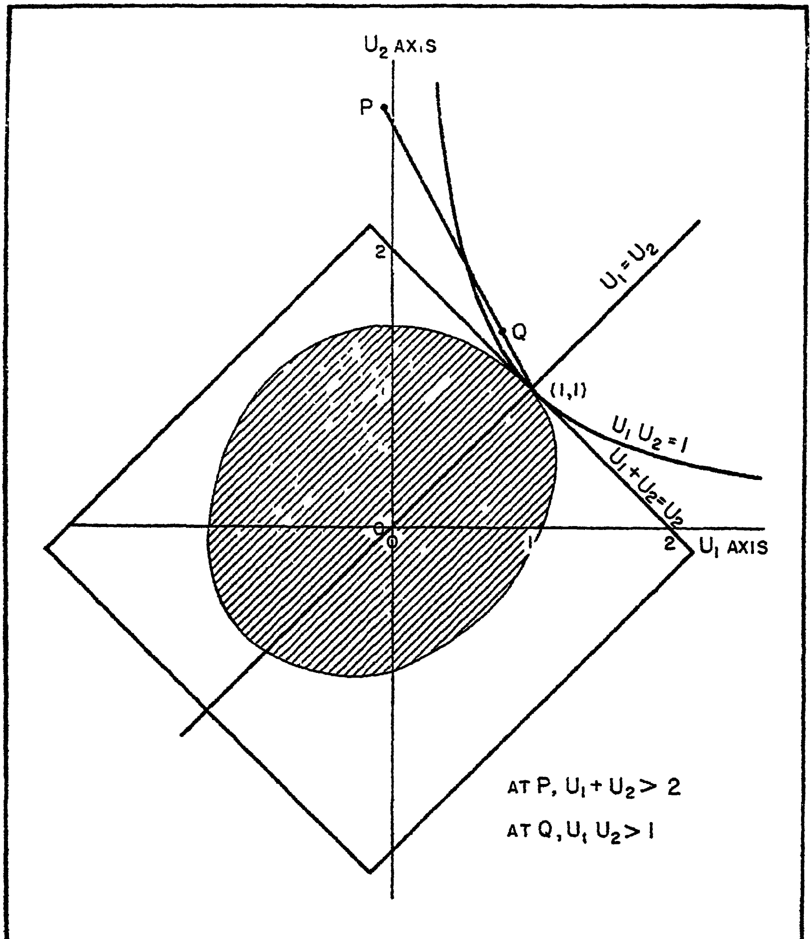

FIGURE 1

to facilitate exchange. Further, let us assume for simplicity that the utility to either individual of a portion of the total number of goods involved is the sum of the utilities to him of the individual goods in that portion. We give below a table of goods possessed by each individual with the utility of each to each individual. The utility functions used for the two individuals are, of course, to be regarded as arbitrary.

Bill's goods	*Utility to Bill*	*Utility to Jack*
book	2	4
whip	2	2
ball	2	1
bat	2	2
box	4	1
Jack's goods		
pen	10	1
toy	4	1
knife	6	2
hat	2	2

The graph for this bargaining situation is included as an illustration (Figure 2). It turns out to be a convex polygon in which the point where the product of the utility gains is maximized is at a vertex and where there is but one corresponding anticipation. This is:

Bill gives Jack: book, whip, ball, and bat,
Jack gives Bill: pen, toy, and knife.

When the bargainers have a common medium of exchange the problem may take on an especially simple form. In many cases the money equiva-

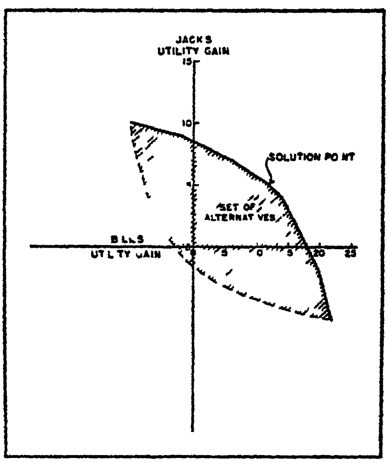

FIGURE 2

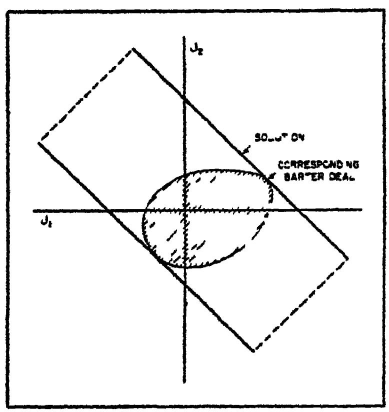

FIGURE 3

FIGURE 2—The solution point is on a rectangular hyperbola lying in the first quadrant and touching the set of alternatives at but one point.

FIGURE 3—The inner area represents the bargains possible without the use of money. The area between parallel lines represents the possibilities allowing the use of money. Utility and gain measured by money are here equated for small amounts of money. The solution must be formed using a barter-type bargain for which $u_1 + u_2$ is at a maximum and using also an exchange of money.

lent of a good will serve as a satisfactory approximate utility function. (By the money equivalent is meant the amount of money which is just as desirable as the good to the individual with whom we are concerned.) This occurs when the utility of an amount of money is approximately a linear function of the amount in the range of amounts concerned in the situation. When we may use a common medium of exchange for the utility function for each individual the set of points in the graph is such that that portion of it in the first quadrant forms an isosceles right triangle. Hence the solution has each bargainer getting the same money profit (see Figure 3).

Princeton University

THE ANALYSIS OF OUTPUT UNDER DISCRIMINATION[1]

By Edgar O. Edwards

"Adjusted concavity" of demand curves as a criterion for comparing a firm's output under discriminatory and simple monopoly pricing is examined. An alternative criterion, the "slope ratio," is developed and its relation to the problem of output change under discriminatory pricing is explained. The relationship of the two criteria is set forth and a graphical presentation of both is given for comparative purposes. Finally, a simple illustrative problem is attacked using the two alternative criteria.

SUMMARY

A theoretical exploration of the problems created by discriminatory pricing involves a determination of the effect of this pricing system on the output of a firm as compared with the output achieved under simple monopoly pricing. Under which pricing system will a larger output be sold?

The concept of "adjusted concavity" presented by Mrs. Robinson[2] was meant to clarify this problem and to provide a criterion for its solution. Its usefulness has been handicapped, however, by the fact that the concept has been explained only in mathematical terms with little or no use of visual or verbal methods. It is possible, however, to derive an alternative criterion that can be as easily explained graphically and verbally as mathematically.

If the total output sold at discriminatory prices in two markets is to be different from that sold at a simple monopoly price, the increase in the output sold in one market must not be equal to the decrease in output in the other market. Thus, the problem resolves itself into determining the *relative* sizes of these changes. For there to be a change in the quantity sold in any *one* market, the marginal revenue derived from the quantity sold in that market at the simple monopoly price must be different from the marginal cost of the *total* output sold at that price. The institution of discriminatory pricing will effect an equation of the marginal revenue derived in each market to marginal cost as well as to each other.

Under certain simplifying assumptions it can be shown that the size of the change in output in each market will depend on (1) the gap, at the simple monopoly price, between the marginal revenue in that market and marginal cost and (2) the rate at which the marginal revenue curve changes with changes in quantity (its slope).

[1] I am indebted to Professor Fritz Machlup who turned my attention to the problem and made valuable suggestions for its solution.

[2] Joan Robinson, *The Economics of Imperfect Competition*, London: Macmillan and Co., 1933, 352 pp.

Bearing in mind that the relative sizes of the changes in output are the sole concern in this problem, these two determining factors can be resolved into a simple "slope ratio," dp/dm, and a comparison of the value of the slope ratio in one market with its value in another can determine, along with the elasticities of demand, whether total output will increase or decrease as a result of the substitution of discriminatory pricing for simple monopoly pricing.

The slope ratio bears a simple relationship to adjusted concavity and both can be measured and applied graphically. However, the directness with which the slope ratio can be derived and used should make the teaching, the understanding, and the use of this part of the theory of price discrimination a great deal easier with no loss of precision.

INTRODUCTION

Mrs. Robinson included in her book a detailed exposition on the analysis of problems involving price discrimination. This presentation marked a great advance toward a clear understanding of such problems, particularly where the question is raised as to the effect on the volume of production. In deciding in what direction total output will change when a system of discriminatory prices replaces a simple monopoly price, Mrs. Robinson leans heavily on the concept of adjusted concavity.[3] This concept, for whose mathematical derivation Mrs. Robinson gives credit to Mr. R. F. Kahn, is not presented graphically nor is it clarified by words in Mrs. Robinson's book. Yet it is the key to an understanding of that pricing method which will yield the larger output under any given set of circumstances.

The concept of adjusted concavity is important primarily in problems involving price discrimination between two or more independent markets. Mrs. Robinson has shown that "total output under discrimination will be greater or less than under simple monopoly according as the more elastic of the demand curves in the separate markets is more or less concave than the less elastic demand curve."[4] A firm which has sufficient monopoly power and can sell a product in different and segregated markets is in a position to practice price discrimination. The problem of how the output of a firm would be affected when, instead of a simple monopoly price, the firm charges a pair (or set) of discriminatory prices is important to society as well as the individual firm. A better understanding of the criterion advanced by Mrs. Robinson should permit specific problems of price discrimination to be more easily and clearly analyzed.

[3] *Ibid.*, p. 40, footnote 3.
[4] *Ibid.*, p. 190. The theorem is analytically proven in footnote 2, p. 193.

In this note an attempt will be made to clarify the concept of adjusted concavity by (1) reviewing Mrs. Robinson's exposition of it, (2) independently deriving a criterion applicable to this price discrimination problem, (3) reconciling this criterion with adjusted concavity, and (4) presenting a simple method of graphical measurement.

We shall find it easier to deal with the concept if a few limiting assumptions are made. These assumptions are:

(a) All demand curves are everywhere continuous and differentiable.

(b) Any demand curve which is concave (or straight or convex) remains so throughout the relevant range.

If either or both of these assumptions are removed, adjusted concavity is not a reliable tool of analysis. For simplicity in presentation we shall also assume that there are two separate and independent markets and that the total revenue curve in each market has one, and only one, maximum.

THE CONCEPT AS DEVELOPED BY MRS. ROBINSON

Attention must first be directed to the original exposition of adjusted concavity. Mrs. Robinson advances as her only word picture of the concept the following explanation:[5]

> The relevant property of the curve, which makes it in this sense more or less "concave," is the rate of change of the slope (at the simple monopoly price) multiplied by the elasticity (at the simple monopoly price, multiplied by the square of the simple monopolist's output in the separate market.

If $p = f(x)$ is an equation of demand and c denotes adjusted concavity, then the elasticity of demand is $-f(x)/xf'(x)$ and Mrs. Robinson's concept can be stated mathematically as follows:[6]

$$(1) \qquad c = -f(x)\,\frac{xf''(x)}{f'(x)}.$$

This equation is not identical with Mrs. Robinson's earlier definition that "$xf''(x)/f'(x)$ may be regarded as a measure of the *adjusted concavity* of the average curve."[7] Here

$$(2) \qquad c = \frac{xf''(x)}{f'(x)}.$$

[5] *Ibid.*, p. 193.

[6] *Ibid.*, p. 193, footnote 2. It should be noted that Mrs. Robinson prefers to treat elasticity of demand, ϵ, as positive.

[7] *Ibid.*, pp. 40–41, footnote 3.

The values for adjusted concavity obtained from these two definitions will always be opposite in sign; and, unless the curve is a straight line so that $f''(x) = 0$, they will be numerically equal only where $f(x) = 1$. As Mrs. Robinson uses them—to compare the adjusted concavities of two curves at the same price—both concepts are equally efficient. However, as equation (2) is mathematically a simpler equation and as it is more easily reconciled with the concept developed below, we shall accept it as the mathematical definition of adjusted concavity.

MODIFICATION OF THE CONCEPT

While it is possible to derive directly from the mathematical definition of adjusted concavity a ratio that can be graphically applied, the independent development of such a ratio should aid in understanding both its usefulness and its limitations.

Figure 1 is a graphical representation of the effect on output in each market when a firm decides to charge a separate price to each market rather than one simple monopoly price to both.[8] The demand curve in market I is designated by D_I, that in market II by D_{II}, and the curve obtained by summing the individual demand curves horizontally (the aggregate demand curve) is designated by D_A. The price charged under simple monopoly, $XP = OL$, is found by locating the point of intersection, K, of marginal revenue in the aggregate market to marginal cost. Marginal cost, $M.C.$, is here assumed to be constant for expositional purposes. [Rising or falling marginal costs would not affect the *direction* of the change in total output (should a change occur) but only the *magnitude* of such a change.][9] We shall also assume that tangents drawn to the marginal revenue curves at points R and T will, within the relevant ranges, approximate the curves to which they are drawn.[10] We wish to find a simple criterion for deciding whether, when price discrimination replaces a simple monopoly price, the increase in output in the more elastic market (market I) is greater or smaller than the decrease in output in the less elastic market (market II.)[11]

The size of the increase in output in market I, (AB), depends upon (a) the amount by which the value of marginal revenue in market I must decrease before equaling marginal cost, i.e., the perpendicular distance RA, and (b) the rate at which it decreases, i.e., the cot α,

[8] The graph is similar to one found in Robinson, p. 191.

[9] See Robinson, pp. 194 and 195.

[10] In essence, this assumption implies that the distances, AR and FT, in Figure 1 are very small and therefore that the demand elasticities of the two curves are very nearly equal. This assumption, made also by Mrs. Robinson (p. 193, footnote 2), is further examined in footnote 15 of this paper.

[11] The elasticities can be quickly judged by comparing the ratios, $p/(p - m)$.

the reciprocal of the absolute value of the slope of the marginal revenue curve in that market. We have, therefore,

$$AB = RA\ (\cot\alpha) = -RA \cdot (dx/dm)_I\,, \tag{3}$$

where $(dx/dm)_I$ denotes the reciprocal of the slope of the marginal revenue curve in market I at the quantity sold in that market under the

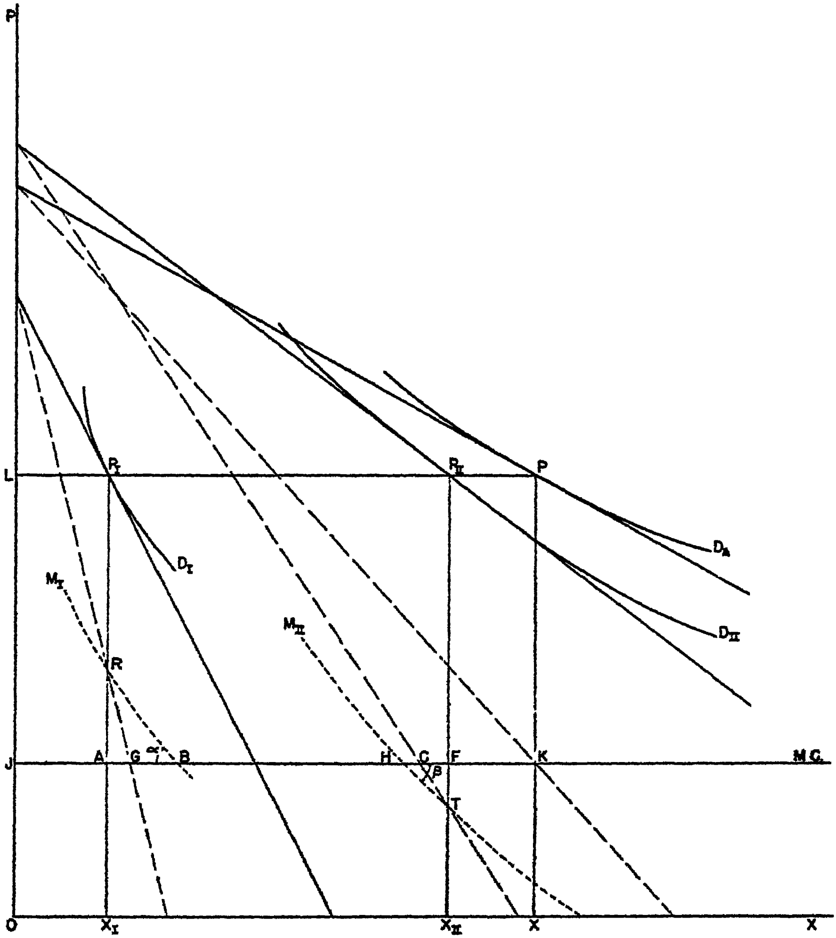

FIGURE 1

simple monopoly price. It can similarly be shown that the size of the decrease in output in market II depends on the necessary increase in marginal revenue and the rate at which it increases. Thus,

$$HF = TF\ (\cot\beta) = -TF \cdot (dx/dm)_{II}\,. \tag{4}$$

Hence, the size of the increase (decrease) in the more elastic market (less elastic market) is *larger* (a) the *larger* the gap, at the simple monopoly price, between marginal revenue in the individual market and simple monopoly marginal cost, and (b) the *smaller* (numerically) the slope of the marginal revenue curve at the quantity sold in the individual market at the simple monopoly price.

In order to simplify equation (3) we may eliminate the term RA. Since $RA/2AG = RP_I/LP_I = -(dp/dx)_I$, we have $RA = -2AG \cdot (dp/dx)_I$. Substituting this value in equation (3), we have

$$AB = 2AG \cdot (dp/dm)_I. \tag{5}$$

Similarly, simplifying equation (4) we obtain

$$HF = 2CF \cdot (dp/dm)_{II}. \tag{6}$$

As the distances AG and CF are equal,[12] the ratio of AB to HF, which determines the relative changes in output, can be written

$$\frac{AB}{HF} = \frac{(dp/dm)_I}{(dp/dm)_{II}}. \tag{7}$$

Therefore, when $(dp/dm)_I > (dp/dm)_{II}$ at the simple monopoly price and market I is the more elastic market, the increase in output (AB) in that market will be greater than the decrease in output (HF) in the less elastic market, and total output will increase with price discrimination. The ratio, dp/dm, which can be called the "slope ratio,"[13] can be used in lieu of adjusted concavity to determine the relative changes in output in separate markets when discriminatory prices are charged rather than a simple monopoly price.

In terms of the slope ratio our criterion becomes: Total output under discrimination will be greater (less) than under simple monopoly if the slope ratio at the simple monopoly price in the more elastic market is greater (less) than that in the less elastic market.

THE SLOPE RATIO AND ADJUSTED CONCAVITY

What is the relationship of the slope ratio, dp/dm, to adjusted concavity, $xf''(x)/f'(x)$? The slope ratio, dp/dm, can be written $(dp/dx) \cdot (dx/dm)$. It is the slope of the demand curve divided by the slope of the marginal revenue curve. We have, therefore,

[12] For proof see Robinson, p. 191 f. Briefly, in Figure 1, $LP_I + LP_{II} = LP$ by definition. Also $JG + JC = JK$. But $JK = IP$. Therefore $AG = CF$.

[13] I am indebted to Professor Fritz Machlup who has used the term "slope ratio" in an unpublished manuscript.

$$\frac{dp}{dm} = \frac{f'(x)}{xf''(x) + 2f'(x)} = \frac{1}{\frac{xf''(x)}{f'(x)} + 2},$$

and

$$\frac{dp}{dm} = \frac{1}{c + 2}. \tag{8}$$

It will be noted that these two terms move in opposite directions. Mrs. Robinson's "more concave" means that the value of adjusted concavity is algebraically *less*. The slope ratio, however, is algebraically *greater* the "more concave" the curve. However, as both measures, the slope ratio and the adjusted concavity, contain the same combination of variables, they are equally efficient when used to determine the relative sizes of changes in output caused by charging discriminatory prices instead of a simple monopoly price.

GRAPHICAL MEASUREMENT

Can the slope ratio, dp/dm, be easily measured graphically? Figure 2 contains a demand curve and its marginal revenue curve with lines drawn tangent to each at the price OL which is assumed to be the simple monopoly price in problems of price discrimination. We shall first prove that our modified measure, the slope ratio, is represented graphically by the ratio AL/AB in Figure 2.

Designating the slope ratio by s, we have

$$s = \frac{\text{slope of demand curve}}{\text{slope of marginal revenue curve}} = \frac{dp}{dx}\frac{dx}{dm}.$$

Now $dp/dx = CL/LP$ at the price OL, and $dx/dm = AM/AB$. But $CL = PM = AL$ and $LP = AM$, so $CL/LP = AL/AM$. Hence

$$\frac{dp}{dm} = \frac{AL}{AB}. \tag{9}$$

This ratio is easily applicable to any graphical problem.

It remains to show that a graphical ratio is also available as a measure of adjusted concavity. As was shown before, $dp/dm = 1/(c + 2)$. Solving for c, and letting $s = dp/dm$, $c = (1/s) - 2$. As $dm/dp = 1/s = AB/AL$, $c = AB/AL - 2$. But as AC/AL is always equal to 2, we can write $c = (AB - AC)/AL = -BC/AL$. In general, then,

$$c = \frac{BC}{AL}, \tag{10}$$

where the value is negative when B lies below C (as in Figure 2), zero when B and C coincide, and positive when B lies above C. In contrast, the slope ratio is normally positive, its value being greater than $\frac{1}{2}$ for concave curves, equal to $\frac{1}{2}$ for straight lines, and less than $\frac{1}{2}$ for convex curves.[14]

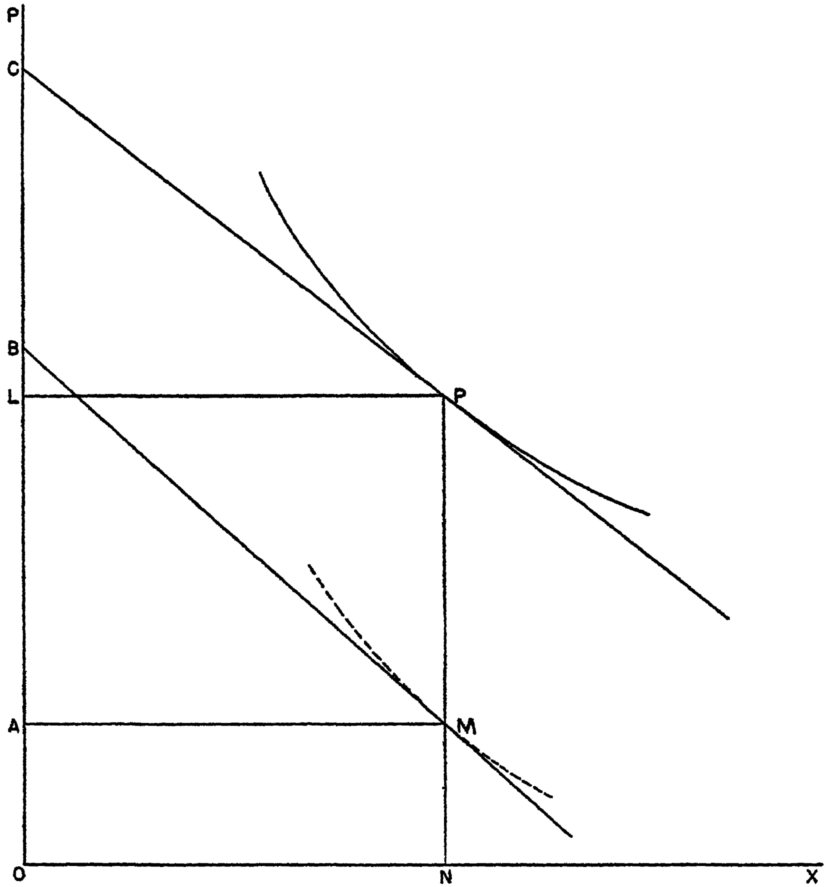

FIGURE 2

Figure 3 illustrates the application of the two ratios to different types of curves. Four markets are shown and in each the simple monopoly price, OL, is the same. Therefore, any two of the markets can be assumed to be individual markets in a price discrimination problem. The

[14] In applying adjusted concavity graphically it must be remembered that Mrs. Robinson's "more concave" means that the value of adjusted concavity is algebraically *less*.

values of the slope ratio, designated by s, and adjusted concavity, c, are shown in each market.

The elasticities of demand in markets *II* and *III* are equal at the simple monopoly price [the ratio, $p/(p - m)$, has the same value in

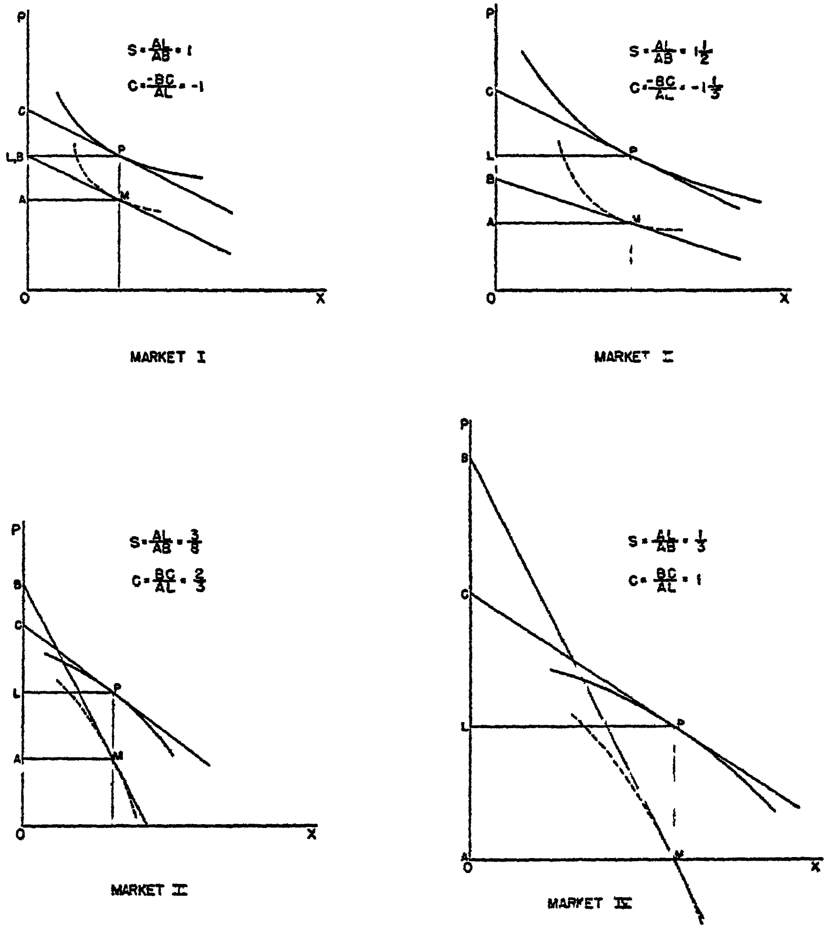

FIGURE 3

each market at the simple monopoly price] so when these two markets comprise the total sales possibilities of the firm, total output will remain unchanged should the firm decide to discriminate. Nor will the prices charged or the quantities sold in the individual markets change, for the marginal revenue in each market is equal even under simple monopoly pricing to marginal cost. The difference between the concavities of the two curves is therefore unimportant.

If we compare market I and market III (or any concave curve with a convex curve), the concavity of the demand curve of market I is seen to be greater than that of market III because $s_I > s_{III}$ (or $c_I < c_{III}$). If the more concave curve is also the more elastic (as in our example), total output will be increased under price discrimination.

Using the values of the slope ratios, market I can be compared with market II. It is seen that total output will decrease under price discrimination for while the demand curve in market I is more elastic [the ratio, $p/(p - m)$, at the simple monopoly price is greater in market I than in market II] it is also less concave ($s_I < s_{II}$). Markets III and IV can be similarly analyzed.[15]

The Johns Hopkins University

[15] The assumption referred to in footnote 10 that the marginal revenue curves can be approximated by their tangents requires further consideration for it is not reasonable to assume that the changes considered here will be truly infinitesimal.

If both marginal revenue curves are concave from above, the approximation used will understate the increase in output in one market and overstate the decrease in output in the other market. Therefore, if according to the comparisons of the elasticities and slope ratios total output should increase under discrimination, we can be confident that the assumption discussed here will not cause error. If, however, the comparisons tell us output should decrease, the correctness of this statement is less certain the larger the respective changes in output. Similar reasoning can be applied to the case where both marginal revenue curves are convex from above.

Where the curves have opposite concavities the straight line approximations cause no error. (See Robinson pp. 192–193 and p. 40.)

REPORT OF THE BOULDER MEETING
AUGUST 29–SEPTEMBER 2, 1949

THE ECONOMETRIC SOCIETY held its American summer meeting at the University of Colorado in Boulder, Colorado, August 29–September 2, 1949, in conjunction with the meetings of the American Mathematical Society, the Mathematical Association of America, and the Institute of Mathematical Statistics. The total number registered was approximately 700. The sessions of the Econometric Society were open to all its members as well as to members of the other organizations and were well attended. Participation in individual sessions ranged from 30 to over 300.

The program was arranged by a committee consisting of George Kuznets, University of California, Berkeley (chairman); R. L. Anderson, University of North Carolina; Andrew T. Court, General Motors Corporation; Morris E. Garnsey, University of Colorado; Nicholas Georgescu-Roegen, Vanderbilt University; Charles Hitch, The RAND Corporation; Lawrence R. Klein, National Bureau of Economic Research; William B. Simpson, Cowles Commission for Research in Economics (ex officio); and Jacob Wolfowitz, Columbia University.

Abstracts of the papers presented at the Boulder meeting are included in the present report and in the *Report of the Symposium on Mathematical Training of Social Scientists* which also appears in this issue. The abstracts are indexed below by speakers in order to facilitate subsequent reference.

CONTRIBUTED PAPERS

Morning and afternoon sessions on Monday, August 29, were devoted to contributed papers on econometrics. Harold Hotelling of the University of North Carolina served as chairman for the morning session and Harold T. Davis of Northwestern University served as chairman in the afternoon.

Utility Analysis of Decisions Affecting Future Well-Being, TJALLING C. KOOPMANS, Cowles Commission for Research in Economics and The University of Chicago.

UTILITY analysis of consumer's choice is based on a complete ordering (admitting indifference as well as preference) of the objects of choice. In static analysis the objects of choice are bundles of commodity flows, that is, vectors x of which the nonnegative components $x_1, x_2, \cdots, x_n, \cdots, x_N$, are rates of consumption of specific commodities, supposed constant over an indefinite period. The intent of this analysis is to treat *preference* and *opportunity* as separate data of the choice problem. The opportunity is defined as a set Q of bundles x given as being accessible to the choosing individual. Best choice is confined to that subset Q_0 of Q (which often is a single bundle x_0 of Q) such that choice between any two bundles x, y in Q_0 is indifferent, while no bundle z in Q outside Q_0 is preferred to any bundle in Q_0. The opportunity set is a linear subspace if the individual buys at constant prices from a given income.

A frequently discussed dynamic generalization of this analysis is obtained by adopting as the objects of preference ranking sequences $X = \{x(1), x(2), \cdots, x(T)\}$ of bundles $x(t)$ of flows which remain constant only within each future time segment (for each value of t) considered. This formulation of the object disregards the desire for post-

ponement of decisions not yet called for by circumstances. It burdens the analysis with an amount of detail regarding alternatives in a distant future such as never sways the decisions economic analysis is concerned with. Furthermore, since it does not allow consideration of assets, the end-of-period position (at time $t = T + 1$) cannot be taken into account, and the arbitrary length T of the period considered will affect the choice which is predicted by the analysis.

These objections can be overcome by adopting as the objects of preference ranking sets S of sequences X of bundles $x(t)$ of flows $x_n(t)$ with the following understanding regarding later choice within the set. A set S of such sequences is called t-uniform if the values $x(1), x(2), \cdots, x(t)$ of the first t vectors are the same for all sequences X in S. We apply the preference ranking only to 1-uniform sets S_1. Any such set S_1 can be exhaustively decomposed into nonoverlapping subsets S_2 which are 2-uniform, by combining into the same S_2 all those sequences for which $x(2)$ has some given value. The individual knows that his choice of S_1 at time $t = 1$ commits him to make a further choice of a 2-uniform subset S_2 of S_1 at time $t = 2$ (if there is more than one S_2 in S_1). S_2 again partitions into 3-uniform subsets S_3, between which further choice is required at time $t = 3$, etc. The preferability of postponement of choice is expressed by the following postulate: If $S_t = S_t' + S_t''$ is t-uniform for some $t \geqq 1$, then the individual will rank S_t at least as high as, and often higher than, either S_t' or S_t''.

While in principle all 1-uniform sets S_1 are assumed subject to preference ranking, the sets contained in the opportunity set Q of sets S_1 may be thought of as given by market anticipations, in the simplest case by single-valued anticipations of future commodity prices and of incomes derivable from specific occupational efforts. Assets can then be entered in the utility function as representative of the sets of consumption sequences they give access to, through direct enjoyment, resale and purchase of other assets or consumption flows, alternatively or in succession, subject to later decision. By introducing end-of-period assets as representing the anticipated level of well-being for $t \geqq T + 1$ for the individual or his heirs, the designation of the period of analysis T represents only a decision of the analyst as to how much of future consumption to introduce explicitly, how much to leave implicit in assets—and thus need not affect the choice predicted by the analysis.

Consumer financing adds to well-being because it gives access to the flows of services associated with possession of durable consumers' goods at an earlier time than would otherwise be possible at the same rate of saving. Compulsory pension plans diminish welfare to the extent that they prohibit the use of accumulated savings (while safeguarding principal by proper depreciation allowances) for gaining access to the services of durable consumers' goods.

Optimal Investment of a Firm, JACOB MARSCHAK, Cowles Commission for Research in Economics and The University of Chicago.

NOTATIONS, DEFINITIONS, ASSUMPTIONS. 1. Use Latin capitals for random variables. Denote the vector of *assets* held by a firm by $a \equiv [a_g] \equiv (a_1, \cdots, a_n)$, and its *profit* by $Y \equiv Y(a)$. Then, at $a = 0$, $Y = 0$; and write $\partial Y/\partial a_g \equiv Y_g$, $\partial^2 Y/\partial a_g \partial a_h \equiv Y_{gh}$, $(g, h = 1, \cdots, n)$. Expand $Y = \sum a_g Y_g + \frac{1}{2}\sum\sum a_g a_h Y_{gh} + \cdots$. Write $\mathcal{E}Y \equiv m_Y$; $\mathcal{E}(Y - \mu_Y)^2 \equiv m_{YY}$; $\mathcal{E}Y_g \equiv q_g \equiv$ the mean of the *marginal profitability* of the gth asset; its variance and covariances are $\mathcal{E}(Y_g - q_g)(Y_h - q_h) \equiv q_{gh}$, $(g, h = 1, \cdots, n)$.

2. Assume that *complementarity* $Y_{gh} = \text{const.} \equiv y_{gh}$, $(g, h = 1, \cdots, n)$. Then the profit function $Y(a)$ is approximately characterized by the vector $[q_g]$ and the two matrices $\| q_{gh} \|$, $\| y_{gh} \|$.

3. *Effective utility function* $u(Y) \equiv u_2[u_1(Y)]$ where $u_1(Y) \equiv$ profit after taxes, and u_2 is the subjective utility function.

4. Expand $v \equiv \mathcal{E}u(Y) = u(m_Y) + u''(m_Y)\cdot m_{YY}/2 + \cdots$. Hence approximately, for v given, $dm_Y/dm_{YY} = -\frac{1}{2}u''(m_Y)/u'(m_Y) \equiv \rho \equiv$ *risk aversion*. (If $u = u_1$, ρ = *tax progressivity;* if $u = u_2$, 2ρ = *flexibility of marginal utility of profit, multiplied by profit.*)

5. *Rational behavior:* maximize v subject to constraints such as (in a simple case) $\sum a_g p_g = k$ (borrowing limit), where p_g are constant prices.

PROBLEM I. Find desirable properties of assets. $\rho > 0$, and, by (5), p_g is proportional to $\partial v/\partial a_g$ and therefore, by (4), also to the quantity $(q_g + \sum y_{gh} a_h - 2\rho \sum q_{gh} a_h)$. Hence high values are desired for q_g, y_{gh}, y_{gg}, $-q_{gh}$, $-q_{gg}$. Examples: comparisons between insurance companies and specialized and diversified investment trusts and producers.

PROBLEM II. Let $X \equiv X(a) \equiv$ the firm's physical output. In the national interest, maximize $\mathcal{E}X$ by choosing an appropriate profit-after-tax schedule, $u_1(Y)$, provided the expectation of tax revenue has a fixed level, c.

Simplifying assumptions: $u = u_1$ for any Y; $n = 2$; $0 \leqslant a_1$ = risky plant; $0 \leqslant a_2$ = riskless bonds. $X = X(a_1)$, $X'(a_1) > 0$; $Y = a_1 Y_1 + a_1^2 y_{11}/2 + a_2 y_2$ approximately, where y_{11}, y_2, and the mean and variance of Y_1 are known. Put $p_1 = p_2 = 1$. Then by (5),

$$(\alpha) \qquad a_1 \leqslant k,$$

$$(\beta) \qquad \mathcal{E}u[Y(a_1)] \geqslant \mathcal{E}u[Y(a_1^*)] \text{ for any } a_1^*,\ 0 \leqslant a_1^* \leqslant k.$$

To find a function u that maximizes a_1 (and hence X), subject to (α), (β), and subject to

$$(\gamma) \qquad \mathcal{E}[Y - u(Y)] = c,$$

is a problem in the calculus of variations. A solution was given for a quadratic tax schedule admitting negative taxes, and with $u(0) = 0$.

Instead, some or all of the following conventional constraints upon the tax schedule might be used: for $Y > 0$, $u \leqslant Y$, $0 < u' < 1$, $u'' < 0$; and for $Y \leqslant 0$, $u(Y) = Y$.

PROBLEM III. Let $Y^{(\tau)} \equiv$ *sequence of profits* in τ years, 0 through $\tau - 1$. Generalize Problem I by substituting $Y^{(t)}$ for Y (t = "horizon"), and by redefining a as a matrix of *decision functions* ("strategy") $a \equiv \| a_g^{(\tau)}[Y^{(\tau)}] \|$, where $g = 1, \cdots, n$; $\tau = 1, \cdots, t$. The best value of a will depend on parameters $[q_g]$, q_{gh}, y_{gh} [defined in (2) and properly generalized] and on additional asset properties, viz. the mutual conversion costs (illiquidities).

PROBLEM IV. Same as Problem III, but the parameters just mentioned are not known in advance. The materialized sequence $Y^{(\tau)}$ serves as a statistical sample of growing size.

The Producer's Cost Function, RONALD W. SHEPHARD, New York University.

G. C. EVANS[1] has made certain interesting dynamic studies of the profit maximization of a single producer and C. F. Roos[2] extended this formulation for the competition of several producers. For these analyses the economics of production is described by a cost function that is taken to depend only on the rate of output. Generally, the cost function conceivably depends also upon the prices of the factors of production, and its definition should be consistent with a formulation in terms of production functions. This consideration suggests a mathematical study of the relationship between cost and production function.

The production function is taken of the form

$$(1) \qquad U = \Phi(x_1, x_2, \cdots, x_n, z_1, z_2, \cdots, z_l),$$

where U is output per unit time and x_i, z_k are independent amounts per unit time of capital and labor factors, respectively. On the basis of a heuristic principle that at any time t for arbitrary U and prices p_i, w_k the quantities x_i, z_k are adjusted so that the cost rate

$$(2) \qquad q = \sum_1^n p_i x_i + \sum_1^l w_k z_k$$

is a minimum, the producer's cost function is defined to be of the form $q = q(U, p_1, \cdots, p_n, w_1, \cdots, w_l)$ by the relations

$$(3) \qquad \lambda = \frac{p_1}{\Phi_1} = \cdots = \frac{p_n}{\Phi_n} = \frac{w_1}{\Phi_{n+1}} = \cdots = \frac{w_l}{\Phi_{n+l}}$$

[1] Griffith C. Evans, *Mathematical Introduction To Economics*, New York: McGraw-Hill, 1930, 177 pp., and "The Dynamics of Monopoly," *American Mathematical Monthly*, Vol. 31, February, 1924, pp. 77–81.

[2] Charles F. Roos, "A Mathematical Theory of Competition," *American Journal of Mathematics*, Vol. 47, July, 1925, pp. 163–175.

for minimum cost together with (1). The mathematical properties of $q(U, p_1, \cdots, p_n, w_1, \cdots, w_l)$ are investigated and a dualistic relation is found between cost and production function. One may be derived from the other by polar reciprocal transformation, a relationship suggested by the notions of support plane, support function, and distance function in the theory of convex bodies. This mathematical relationship is particularly important for statistical studies of cost and production function.

A class of production functions called homothetic, defined by

$$U = \Phi[\sigma(x_1, \cdots, x_n, z_1, \cdots, z_l)] \quad \text{or } f(U) = \sigma(x_1, \cdots, x_n, z_1, \cdots, z_l), \tag{4}$$

where σ is a scale function, homogeneous of degree one, is considered. It is shown that homotheticity is a necessary and sufficient condition for the cost function to be of the form

$$q = f(U)\cdot\lambda(p_1, \cdots, p_n, w_1, \cdots, w_l), \tag{5}$$

where λ is a scale function of the prices of the factors of production. Except for a multiplicative constant, $f(U)$ is Evans' cost function, and in these terms the dynamic studies of Evans and Roos may be generalized in a straightforward way.

The functions λ and σ are shown to be essentially index numbers of price and quantity of the factors of production, and a special representation of them is displayed which is susceptible of easy statistical estimation of the parameters involved and leads to a mathematical definition of the Cobb-Douglas production function for the special case when $f(U) \equiv U$.

In terms of linear constraints upon x_i, z_k the assumption of independent factors may be relaxed and this more general situation reduced precisely to the foregoing analysis by suitable elimination of variables in (2) and redefinition of prices in this reduced form.

The Stationary Theory of the Firm, PAUL W. MCGANN, Operations Evaluation Group, Massachusetts Institute of Technology.

THIS paper was designed primarily to review the types of side relations appropriate for analyzing the theory of the firm by relative maximization under stationary conditions (known patterns of change over time).

The analysis began with the simplest static case and developed appropriate side relations for increasingly complicated situations. The static cases covered were those of multiple outputs, multiple markets, multiple plants, multiple stages of production, multiple production functions, nonprice competition, and nonologopoloid competition. The most in-

teresting feature is the importance of identity side relations in these cases (e.g., the total amount of an output sold equals the sum of the amounts sold in each market).

Stationary analysis involved in addition dynamic identities relating stocks and flows, both of which enter as variables. Types of stock variables treated were receivables, fixed assets, inventories, cash (under no risk), marketable securities, and liabilities. Foregone return, discounting over time, and "maintenance" had to be taken into account.

A brief indication was made of the extension of this rather traditional form of analysis to cases of risk, uncertainty, changing external conditions over time, and oligopoloid competition. Inevitably this involved shifting emphasis from profit maximization to utility maximization by sole proprietorships and to group welfare maximizations by partnerships and corporations. Even briefer remarks were directed toward the problems of incorporating the theory of games into the theory of firms operating under circumstances where the "payoff surfaces" of the players are only roughly estimated and where there is the major problem of estimating what other players' payoff surface estimates are when statistical research is costly and external conditions continually change the basic theoretical payoff surface.

Risk Allowances for Price Expectations, PAUL B. SIMPSON, Oregon State College. (The complete paper will be published in a subsequent issue.)

Methods of Solution in Game Theory, MELVIN DRESHER, The RAND Corporation.

CONSIDER the two-person zero-sum game, with a finite number of strategies, described by the payoff matrix $A = \|a_{ij}\|$, where a_{ij} represents the payment to player I if he chooses strategy $i = 1, 2, \cdots, m$ and player II chooses strategy $j = 1, 2, \cdots, n$. Let the column matrices X, Y represent mixed strategies of players I and II, respectively, where the components x_i, y_j of the respective matrices are the probabilities of the corresponding strategies. Then there exists a pair of mixed strategies X^*, Y^* such that

$$\min_{Y} X^{*\prime}AY = \max_{X} X'AY^* = v,$$

where X' is the transpose of X. X^*, Y^* is said to be a solution of the game having a *value*, v.

If there exists a solution X^*, Y^* such that $AY^* = v1$ and $A'X^* = v1$, where 1 is the column matrix all of whose components are unity, then X^*, Y^* is a *simple solution* of the game. A necessary and sufficient condi-

tion that a game described by a square, nonsingular matrix A have a simple solution is that all the components of

$$X^* = \frac{(A')^{-1}1}{l'(A')^{-1}1}, \qquad Y^* = \frac{A^{-1}1}{l'A^{-1}1}$$

be nonnegative. Thus, if a game has a simple solution, it is readily obtained.

If a game does not have a simple solution, it can nevertheless be solved by examining square submatrices for simple solutions. Every solution of a game A is a convex linear combination of a finite number of *basic solutions* and every basic solution is derived from a simple solution of some square nonsingular submatrix of A. Thus to obtain all basic solutions we need to examine each square nonsingular submatrix of A for a simple solution. Having obtained a simple solution of the submatrix, we introduce zero components for the remaining rows and columns of A, and test whether the full vectors X, Y solve A, i.e., whether

$$\max_i \sum_{j=1}^{n} a_{ij} y_j = \min_j \sum_{i=1}^{m} a_{ij} x_i .$$

If X, Y solves A then it is a solution of A and the value of the game is $\max_i \sum_{j=1}^{n} a_{ij} y_j$.

It is frequently possible to reduce the size of the game matrix by simple dominance considerations. If a row is dominated by some other row or by some convex linear combination of rows, then the dominated row may be eliminated from the matrix. If a row dominates some convex linear combination of other rows, then some one of them may be eliminated. Similarly, we may eliminate columns. In general, reduction by dominance may lose some solutions. However, reduction by strict dominance retains all solutions.

If a game matrix is m by 2, it can be solved graphically in two dimensions. Using triaxial coordinates we can solve an m by 3 game matrix graphically and also in two dimensions. If the game matrix is very large it is possible to solve it by an iterative process which requires only addition and the location of maxima and minima—at each step the method chooses for each player a strategy which is best against the opponent's mixture cumulated to date.

The solution of games having a continuum of strategies is very difficult and methods exist only for a few special classes of payoff functions. If the payoff, $M(x, y)$, is continuous and convex in y for each x, then the value of the game is $v = \min_y \max_x M(x, y)$. Player II has pure strategies which are optimal—every y which minimizes $\max_x M(x, y)$. Player I has optimal strategies which are generally mixed—they make use of all X

such that $M(X, Y) = v$, where Y is any pure strategy which is optimal for player II.

If the payoff $M(x, y)$ is a polynomial in each variable then it is possible to reduce the solution of the game problem to the solution of certain systems of equations—linear in some cases, nonlinear in the remaining.

SYMPOSIUM ON MATHEMATICAL TRAINING FOR SOCIAL SCIENTISTS

On Tuesday morning, August 30, a symposium was held jointly with the Institute of Mathematical Statistics and the Mathematical Association of America on the general topic of mathematical training for social scientists. Jacob Marschak of the Cowles Commission for Research in Economics served as chairman. A report of this symposium appears elsewhere in this issue.

STATISTICAL INFERENCE IN DECISION MAKING

Sessions on statistical inference in decision making were held jointly with the Institute of Mathematical Statistics on Tuesday afternoon and Wednesday morning, August 30 and 31. The chairmen of the meetings were Jerzy Neyman of the University of California and Abraham Wald of Columbia University, respectively. The following papers were presented:

Decision Functions, Aryeh Dvoretzky, Hebrew University of Jerusalem and the Institute for Advanced Study.

This paper constitutes the introductory address in the sessions on Statistical Inference and Decision Making. Thus its aim is to introduce and explain the main concepts and problems encountered in the modern general theory of decision functions.

After indicating the decision-making character of statistical problems, the main ideas of the general decision theory—as developed primarily by A. Wald—are introduced through the consideration of the following simple problem.

We are given an urn containing (in unspecified numbers) coins of n varieties. The probability of tossing heads with a coin of the ith ($i = 1, 2, \cdots, n$) variety is p_i. A coin is drawn from the urn and tossed k times. On the basis of the outcomes of these tossings it is required to decide to which variety the drawn coin belongs.

In accordance with Wald's concept of weight function, penalties due to incorrect decisions are introduced. In the case treated here these are given by an n by n matrix of nonnegative elements with zeros along the main diagonal. The nonstatistical case $k = 0$ (no observations) is treated first. Here the game-theoretic features of the problem (in the von Neumann sense) are particularly clear. The role of randomized strategies and the minimax principle are discussed and the concepts of expected loss, Bayes solution, admissible solutions, complete class of solutions, minimal solution, and least favorable distribution are introduced. The case $k = 1$ is treated next, and here the preceding concepts with "solution" replaced by "decision-rule" or "decision function" are again encountered. The role played by randomization is shown to become less and less marked as k, the number of observations, increases.

The various concepts introduced above are then interpreted in more general situations. Ultimately, again following Wald, it is shown how to take into account also the cost of experimentation and, indeed, to treat fully the problem of design of experiments. Thus the statistician can grapple not only with the problem of what decision to make on the basis of given data but also with the problem of deciding (also sequentially) about the collecting of these data, i.e., the scheme of experimentation.

Some Recent Results in the Theory of Statistical Decision Functions, ABRAHAM WALD, Columbia University.

AN OUTLINE of the basic ideas and results of a recently developed theory of statistical decisions is given in this paper. Let $X = \{X_i\}$ ($i = 1, 2, \cdots$, ad inf.) be a sequence of chance variables. The joint distribution F of X is assumed to be unknown. It is known, however, that F belongs to a given class Ω of distribution functions. There is a space D given whose elements d represent the possible decisions that can be made by the statistician. By experimentation we mean making observations on the chance variables $X_1, X_2, \cdots$, etc. A decision rule δ is a rule for carrying out experimentation and making a decision d at the termination of experimentation.

There is a weight function $W(F, d)$ given which denotes the loss suffered by the statistician when F is the true distribution of X and the decision d is made. Let $c(x_1, \cdots, x_m)$ denote the cost of experimentation if experimentation consists of m observations and x_i is the observed value of X_i. The expected value of the loss $W(F, d)$ and the expected cost of experimentation depend only on the true distribution F and the decision rule δ adopted by the statistician. Let $r(F, \delta)$ denote the sum of these two expected values. The quantity $r(F, \delta)$ is called the risk when F is true and the decision rule δ is adopted. If δ_1 and δ_2 are two decision rules such that $r(F, \delta_1) \leqq r(F, \delta_2)$ for all F, and $r(F, \delta_1) < r(F, \delta_2)$ for at least one F, we shall say that δ_1 is uniformly better than δ_2.

A certain convergence definition is introduced in the space Δ of all decision rules δ and it is shown that under very mild restrictions the space Δ is compact. The risk function $r(F, \delta)$ is shown to have certain continuity and semicontinuity properties.

Let ξ be a probability measure in Ω (defined over a suitably chosen Borel field). A decision rule δ_0 is said to be a Bayes solution relative to ξ if $\operatorname{Min}_{\delta} \int_{\Omega} r(F, \delta)d\xi = \int_{\Omega} r(F, \delta_0)d\xi$. A decision rule δ_0 is said to be a minimax solution if $\operatorname{Sup}_F r(F, \delta_0) \leq \operatorname{Sup}_F r(F, \delta)$ for all δ. A class C of decision rules δ is said to be complete if for any δ not in C there exists a decision rule δ^* in C such that δ^* is uniformly better than δ.

Existence theorems and various results concerning Bayes and minimax solutions are given. A number of results concerning complete classes of decision rules are also obtained. It is shown, among other things, that under some mild restrictions the class of all Bayes solutions is complete.

Remarks on a Rational Selection of a Decision Function. HERMAN CHERNOFF, University of Illinois.

ACCORDING to the Wald formulation of the theory of decision functions many problems in the theory of statistics reduce to the problem of selecting a strategy or decision function from several available strategies when all that is known of the state of nature is that it is one of a given set of states of nature. For each strategy and each state of nature there corresponds a payoff (in utility). Several criteria which have been tentatively suggested in the past for selecting "good" strategies are seen to have shortcomings. Since a criterion for selecting strategies should have certain properties of rationality and consistency, a list of such properties is set forth as a set of axioms to be obeyed by a "rational criterion." For the case when there are two possible states of nature it is seen that a rational criterion must consist of maximizing the average payoff, i.e., assuming that each unknown state of nature has an a priori probability of one-half of being the true state of nature.

The Role of Personal Probability in Statistics, L. J. SAVAGE, The University of Chicago.

THE BASIC theoretical problem of statistics may be formulated as that of finding satisfactory rules for acting in the face of uncertainty. Modern work on this problem has been strongly conditioned by the tendency of modern statisticians to countenance no other than the frequency definition of probability. While these efforts, culminating in Wald's theory of minimum risk, have in some ways been remarkably successful, they seem ultimately (at least to some) to lead to insurmountable obstacles.

It is the purpose of this paper to suggest that these obstacles may be bypassed by introducing into statistical theory a probability concept,

which seems to have been best expressed by Bruno de Finetti,[1] possibly side by side with the frequency concept. According to de Finetti, plausible assumptions about the behavior of a "reasonable" individual faced with uncertainty about future events imply that he associates numbers with the events, which from the purely mathematical point of view are probabilities. These personal probabilities are in principle measurable by experiments which may be performed on the individual either by himself or by others, and their interpretation is such as to make clear (in view of the von Neumann-Morgenstern theory of utility) how the individual should act in the face of uncertainty.

The theory compares unsatisfactorily with others (in particular Wald's theory of minimum risk) in that it does not predict or, speaking normatively, demand the process of deliberate randomization. It seems to me that both Wald's and de Finetti's theories are incomplete descriptions of what statistical behavior is and should be, and that we may look forward to their unification into a single more satisfactory theory.

Complete Classes of Decision Functions for Some Standard Sequential and Nonsequential Problems, Milton Sobel, Columbia University.

This paper was devoted to the derivation of complete classes of decision procedures for standard sequential and nonsequential problems. The risk function employed in all these problems was a simple one, i.e., the loss is one for a wrong decision and zero for a correct decision and the cost $c[n(x)]$ of taking n observations equals $k \cdot n(x)$ for a given constant k.

For example, in one of the sequential cases concerning a binomial chance variable X such that $P\{X = 1\} = p$ and $P\{X = 0\} = 1 - p$, the problem of testing $H_0{:}p \leqq p_0$ against $H_1{:}p > p_0$ was considered. It is assumed that there is given an open interval $Z{:}(p_1, p_2)$ containing p_0 which acts as an indifference zone, i.e., if the true p is in Z, then either decision, H_0 or H_1, is correct. Let d_n denote the number of ones in n observations. It is shown that every Bayes solution has the following property. For each n there exists an interval $I_n{:}[d_1(n), d_2(n)]$ such that the Bayes procedure is to continue taking observations until d_n falls outside I_n. If $n = M$ is the first time this happens, then we accept H_0 or H_1, respectively, according as $d_M < d_1(M)$ or $d_M > d_2(M)$. The set of all such decision procedures forms a complete class for this problem, i.e., for any decision procedure not in this class there exists another in the class which is uniformly better in the sense of a smaller risk function.

A similar result was shown for a normal chance variable with unknown mean θ and known variance and for a Poisson chance variable with un-

[1] See Bruno de Finetti, "La Prévision: ses lois logiques, ses sources subjectives," *Paris Université Institute Henri Poincaré, Annales*, Vol. 7, 1937, pp. 1-68, and earlier articles by the same author.

known mean θ. In each of these cases the hypothesis tested is $H_0:\theta \leqq \theta_0$ against the alternative $H_1:\theta > \theta_0$. The result is the same as that above except that d_n is replaced by $\bar{x}_n$, the mean of the first n observations.

Several nonsequential problems were also considered.

USE OF CROSS-SECTION DATA

Theodore W. Anderson of Columbia University was the chairman of a session on Tuesday, September 1, devoted to papers on the use of cross-section data. George M. Kuznets of the University of California served as discussant for the papers presented at the session.

Problems and Methods of Sampling for Economic Data, J. F. Daly, Bureau of the Census. (This paper was read by Eli S. Marks.)

Although we cannot, without knowing independently the very number we are trying to estimate, determine the exact error of a result based on a sample, we can often set practical bounds on the error to be expected from a sample survey by using the theory of probability. In order to justify the use of probability theory for measuring the expected accuracy of a sample survey, however, it is not enough merely to assume that the theory applies. Fortunately, it is possible in many important cases to set up a sample survey on the basis of techniques which are known through experience to produce results which obey the theory of probability. Such techniques reduce essentially to selecting cards from a file by tossing a coin, rolling a die, or using a table of random numbers.

Even when samples are drawn from a list by the use of random numbers or by other techniques which are known to obey the laws of probability, the amount of variability to be expected in the characteristics of the various possible samples depends on the size of the samples and on the detailed techniques of drawing them. Thus, for example, if we can divide a list of business establishments into subclasses (strata) in such a way that there is on the average less difference in total sales between establishments in the same class than between establishments in different classes, and if we select independent samples of the proper size from each of these classes or strata, we can make the possible samples drawn in this way have less variability on total sales than samples of the same total size drawn without regard to stratification.

Similarly, if we have data about the universe we may use it in the estimation procedure instead of in the sampling procedure to reduce the variability of the possible sample estimates without destroying their

dependence on the laws of probability. The use of available universe data, either in the sampling process in the form of stratification or in the estimation process in the form of ratio or regression estimates, is particularly important in surveys of economic data, where the variability of more straightforward sample estimates based on a simple random sample is likely to be too great to be tolerated.

In conclusion, a word needs to be said about the notion of sampling bias. The bias of a sample estimate may be defined roughly as the difference between the number being estimated and the average value of the estimate of this number based on a long series of samples drawn under the same conditions. The problem of reducing or eliminating biases arising from errors of measurement is sufficiently important to deserve a separate paper in its own right, and will be left to a later speaker. It should be noted here, however, that a biased estimate may sometimes be more accurate on the average than an unbiased estimate based on the same data, for the relative accuracy of the estimate depends on both sampling variability and bias. If the bias is small compared to the sampling variability, the contribution of the bias to total sampling error is much smaller than the bias itself.

The Use of Survey Data in Econometric Studies, CLIFFORD HILDRETH, Cowles Commission for Research in Economics.

IN ESTIMATING parameters of economic relations there is great need for combining cross-section and time series data. Either type of data by itself has deficiencies, some of which can be lessened by their joint use. Time series data, for example, nearly always provide the investigator with a small sample. Changes of structure in economic models and changes in definitions of tabulated variables insure this. In addition some variables, say general level of education of the public, change so slowly over time that time series data may contain little information about the influence of such variables on economic behavior.

Other variables, say prices of easily transported commodities, would vary little in a cross-section study. Cross-section data are ordinarily collected by surveys which are expensive and through which it is usually difficult to get observations on a complete set of economic variables. Interviews are limited in time and for some variables response error may be large. Thus, a study which made use of both types of data could be expected to yield better estimates of parameters.

In addition to the possibility of using both types of data in a single investigation, there is the possibility of comparing results of cross-section studies with time series results to verify them or improve judgments about the reliability of the results. This raises the question of comparability of results. In the past, it has sometimes been possible to show that

investigators using different types of data have not really estimated the same parameters.

A natural way to insure comparability of results of cross-section and time series studies or to conduct a study using both types of data is to start by constructing an economic model that accounts for the distribution of the endogenous variables both over individuals and over time. If statistical specifications are then derived from the same basic economic model, results should be comparable. A simple example of such a model was presented and sample equations for use in cross-section and time series studies were derived.

Derivation of aggregate equations was simple in the model used because the equations explaining individual behavior were assumed to be linear in the variables. Thus the variables that entered aggregate equations were the sums of the variables in individual equations (i.e., population totals) and these are ordinarily compiled by data-gathering agencies. It was pointed out, however, that other assumptions about the form of the individual equations would lead to aggregate equations containing variables not ordinarily compiled. Klein,[1] for example, considered individual production functions linear in the logarithms of the observed variables and derived aggregate equations in the geometric means of the variables in the individual equations. The speaker considered the case of quadratic functions in the individual equations. These led to the appearance of variances and covariances or sums of squares and cross products in the aggregate equations. These are usually not observable in a time series study but in special cases the investigator might be willing to make special assumptions that would permit the substitution of other variables for the nonobserved variances and covariances.

If the variances and covariances are assumed to be smooth functions of time of specified algebraic form, then functions of time could be substituted in the aggregate equations. To assume that frequency distributions of the individual variables changed from year to year by a scale factor only would make it possible to express variances and covariances in terms of squares and cross products of the observed sums of the individual variables. Economists have sometimes made an assumption of this sort when they have assumed that the same Lorenz curve described the distribution of personal incomes for a number of years. If assumptions such as the above are made, it is desirable to have cross section data for enough time periods to provide at least a rough check of the assumptions. Professor Marschak has pointed out that government agencies frequently estimate population totals from cross-section data and then publish only the estimated totals. Publication of frequency distributions or moments of them would make special assumptions like those above unnecessary.

[1] "Remarks on the Theory of Aggregation," ECONOMETRICA, Vol. 14, October, 1946, pp. 303–312.

Financial Surveys Among Consumers—Their Purposes and Methods, GEORGE KATONA, Survey Research Center, University of Michigan.

THE PAPER deals with three major questions that confront economic surveys. The answers to the questions are derived from the Surveys of Consumer Finances conducted by the Survey Research Center for the Federal Reserve Board.

1. *What kinds of data are collected?* (*a*) Micro-economic financial data, (*b*) data on attitudes toward economic matters, and (*c*) combinations of (*a*) and (*b*).

Answer (*a*) raises the question of aggregates vs. individuals. This leads to the discussion of the central problem of social psychology: only individuals behave and not groups, but individuals behave differently according to the group to which they belong. We are not interested in individuals as such but in homogeneous groups of individuals. Micro-economic data consist of frequency distributions and of characteristics of homogeneous groups.

Answer (*b*) raises the question of measurability. Habits, attitudes, motives, expectations are intervening variables that are not directly observable but are measurable by studying the relation of changes in environment to changes in behavior. In psychology the concept "dynamic" does not mean simply "taking time lags into consideration," but also means considering forces, drives, answers to the question "Why?" Intervening variables may show whether observed correlations reflect cause-effect relationships. The psychological variables are not exogenous.

(*c*) Since it is meaningless to speak of the average motive to save or the average price expectation, the relationship between attitudes and financial data is analyzed by determining the financial position or the behavior of groups that have the same attitudes (e.g., the rate of saving of those who expect their income to increase greatly).

By making one survey, we obtain data concerning the situation at a given time. By making several successive surveys, we obtain time series of (*a*), (*b*), and (*c*).

2. *Why are we interested in these data?* (*a*) To complete the analysis of what has happened and to assess recent past and current development. An increase of the aggregate national income by 10%, for instance, may mean different things and may have different effects according to whether all families, the majority of families, or the minority of families had an income increase. Similar considerations make data on the distribution of incomes, liquid asset holdings, amounts saved, etc., necessary.

(*b*) To study the functional relationship between different variables and to test hypotheses about consumer behavior. It is not enough to tabulate the relationship, for instance, between income changes and saving. The question is, "Under what circumstances does increase in

income and increase in saving, under what circumstances increase in income and decrease in saving, etc., occur?"

3. *What kinds of methods are used?* (*a*) Sampling. Probability sampling was discussed in Mr. Daly's paper. (*b*) Interviewing. Personal interviews; open-ended questions; extensive probing and collecting data on relationships; pitfalls of single direct questions; advantages of obtaining financial and attitudinal data in the same survey for purposes of building rapport. (*c*) Errors. Sampling errors, reporting errors, nonresponse errors, and their relative importance for single surveys and for time series.

* * *

RIETZ MEMORIAL LECTURE

No session of the Econometric Society was scheduled on Thursday afternoon, September 1, in order to permit attendance at the Rietz Memorial Lecture sponsored by the Institute of Mathematical Statistics. Jerzy Neyman of the University of California, Berkeley, discussed *Consistent Estimates of the Linear Structural Relation in the General Case of Identifiability.*

* * *

LINEAR PRODUCTION PLANS

Linear Production Plans was the subject of the concluding session of the Society, held Friday morning, September 2. L. J. Savage of The University of Chicago was chairman and Tjalling C. Koopmans of the Cowles Commission for Research in Economics offered a prepared discussion.

Linear Programming and the Theory of Games, A. W. Tucker, Princeton University.

Solution of a maximum problem in linear programming, such as (1) below, is *dual* to solution of a minimum problem (2), and together they are equivalent to solution of a problem in two-person zero-sum games (3). Thus, given an m by n matrix A, an m-vector b, and n-vector c, and letting δ denote a scalar, x a variable n-vector, u a variable m-vector (vectors being one-column matrices unless transposed by the accent $'$), any one of the following three statements can be shown to imply the other two: (1) δ is the maximum value attained by the linear function $c'x$ subject to the inequalities $Ax \leqq b$, $x \geqq 0$; (2) δ is the minimum value attained by the linear function $b'u$ subject to the inequalities $A'u \geqq c$, $u \geqq 0$; (3) the two-person zero-sum game with the $m + 1$ by $n + 1$ payoff matrix

$$\begin{bmatrix} A & -b \\ -c' & \delta \end{bmatrix}$$

has game-value zero and optimal (or good) mixed strategies for both players in which the pure strategies corresponding to the last row and last column occur with positive probabilities (see von Neumann and Morgenstern, *The Theory of Games and Economic Behavior*).

One generalization is to replace c' by a k by n matrix C, δ by a k-vector d, and a simple maximum by a maximum in the sense of partial order. Then d becomes an "efficient point" in the sense of T. C. Koopmans. Another generalization is to allow the restrictions to consist partly of equalities rather than wholly of inequalities, as with Koopmans' "intermediate products."

Other relations between linear programming and game theory are known. Solution of a game with payoff matrix A and positive game-value corresponds to maximizing the sum of the components of x subject to the inequalities $Ax \leqq 1$, $x \geqq 0$, and minimizing the sum of the components of u subject to the inequalities $A'u \geqq 1$, $u \geqq 0$. Also, G. W. Brown and G. B. Dantzig have found that (1) and (2) above are equivalent to solving a game whose payoff matrix can be arranged in the following skew-symmetric form:

$$\begin{bmatrix} 0 & A & -b \\ -A' & 0 & c \\ b' & -c' & 0 \end{bmatrix}.$$

These matters were discussed at the Linear Programming Conference held by the Cowles Commission at The University of Chicago, June 20–24, 1949. The details are to be published in a forthcoming Cowles Commission Monograph on this subject.

The Stability of Inverses of Input-Output Matrices,[1] OSKAR MORGENSTERN and MAX A. WOODBURY, Princeton University.

INPUT-OUTPUT matrices are based on observations that are naturally afflicted with errors. It is important to investigate quantitatively the extent to which their inverses are affected by variations in the quality of the information. This problem has to be approached empirically at first for large matrices in view of the limited experience that exists in handling such matrices. Therefore, two 18 by 18 input-output matrices have been inverted, the second differing from the first in only 42 places by at most 2 per cent of the respective entries. This percentage is very

[1] Research under contract with Office of Naval Research.

much smaller than the correction for errors that would have to be taken into account, and indications are that the errors vary widely from one field to another. But for the sake of simplification the small, uniform percentage was chosen.

The inverses prove to be highly stable; this is attributed to the fact that the matrices differ but little from the identity matrix. It is an open question whether this fact adequately represents the great interdependency of economic activities, i.e., whether original data of far superior quality would not produce input-output matrices with a much smaller number of zeros. Therefore, new data collections, now in progress, may result in the establishment of matrices of less convenient type.

The stability of inverses must not be interpreted as proving stability of the economy. It is merely an indication how, and to what extent, errors in the given matrix carry over into its inverse. The investigation contributes, therefore, to an evaluation of the prognostic value of input-output tables in linear programming and similar applications.

Dr. Max A. Woodbury has investigated some mathematical aspects of these problems and has determined bounds for the ratio of the norms of changes in the original and inverted matrices. He also has obtained further general results which prove useful in application. A paper, read on his behalf, is abstracted below.

Certain results on the relation of the inverses of two slightly different input-output matrices are studied and explained (an output-input matrix is the inverse of an input-output matrix), viz. the tendency for the inverted matrices to differ in the same places as the original matrices with changes of about equal magnitude but of opposite sign to the original changes. This tendency is a consequence of the fact that input-output matrices and hence their inverses, the output-input matrices, differ only moderately from a unit matrix.

Some related results are also obtained, namely a bound for the ratios of the norms of the changes in the original and inverted matrices (thus related to the stability of the inverses of the matrices) and a method for obtaining the exact difference between the inverse of two matrices that differ only in certain rows. An explicit formula is given for original matrices differing only in one row. Specifically, if a and $'a$ are two matrices and A and $'A$ are their inverses, then

$$\frac{1}{(1 + \sum d_i)(1 + \sum d_j')} \leqslant \frac{N('A - A)}{N('a - a)} \leqslant (1 + \sum D_i)(1 + \sum D_j'),$$

where $N(b) = \sum_{i,j=1}^{n} b_{ij}$ and where

$$D_i = \max_p (|A_{ip} - \delta_{ip}|), \qquad D_j' = \max_q (|'A_{qj} - \delta_{qj}|)$$

and

$$d_i = \max_p (|a_{ip} - \delta_{ip}|), \qquad d'_j = \max_q (|{}'a_{qj} - \delta_{qj}|).$$

Further, if the matrices a and $'a$ differ only in the rth row, then

$$'A_{ij} = A_{ij} - \left(A_{ir} \sum_{k=1}^{n} \Delta a_{rk} A_{kj}\right) \Big/ \left(1 + \sum_{k=1}^{n} \Delta a_{rk} A_{kr}\right),$$

which generalizes the result of Sherman and Morrison (*Annals of Mathematical Statistics*, Abstract, Vol. 20, June, 1949, p. 317).

THE MATHEMATICAL TRAINING OF SOCIAL SCIENTISTS

Report of the Boulder Symposium

A symposium on The Mathematical Training of Social Scientists was held at Boulder, Colorado, on Tuesday, August 30, 1949, as part of the American summer meeting of the Econometric Society. The Institute of Mathematical Statistics and the Mathematical Association of America joined with the Econometric Society in sponsoring this symposium, and the attendance exceeded 300. Jacob Marschak of the Cowles Commission for Research in Economics organized the session and served as its chairman.

At the conclusion of the symposium, the following motion, advanced by George Kuznets of the University of California, Berkeley, was adopted by unanimous vote of those present:

> Members of the Mathematical Association of America, the Institute of Mathematical Statistics, and the Econometric Society, assembled in a joint session in Boulder, Colorado, on August 30, 1949, are of the opinion that officers of these societies should study the need for better mathematical training of social scientists, and the ways and means to improve mathematical preparation of social scientists, and that such a study may be most effectively conducted by a joint committee, possibly in cooperation with other interested societies, and in close touch with the Social Science Research Council, the National Research Council, or other national bodies concerned with general education and research. It is suggested that this committee report the results of its deliberations at the next joint meeting of the original participating societies.

In accordance with the above provisions, the following have been named to a study committee by the original sponsoring organizations: William Madow, University of Illinois (chairman); T. W. Anderson, Jr., Columbia University; Frank L. Griffin, Reed College; Leonid Hurwicz, University of Illinois; Jacob Marschak, Cowles Commission for Research in Economics; E. P. Northup, University of Chicago. Sponsorship of the committee will be broadened by the inclusion of additional organizations, each of which will name members to the committee. It is expected that progress reports of the work of the committee will be made in 1950, and that the final report will be presented at the meetings in September, 1951.

Statements by participants in the symposium, in somewhat abbreviated form, are given below. They are included in the index of the Boulder Meeting appearing on page 173.

The Mathematical Training of Social Scientists, FRANK L. GRIFFIN, Reed College.

THE 1930 committee of the Social Science Research Council[1] construed its problem as somewhat different from, and in a sense broader than, the one which is the subject of this symposium, which, I take it, is the training of professional social scientists. The committee had that in mind, but gave most of its thought to the mathematics needed by students of the social sciences who may take only a few courses in this field. This large and important group of students does not need much technical proficiency in the use of mathematics. They will not be producers of mathematical work in the field and do not need a producer's command of mathematics. They do, however, need to be competent consumers.

By surveying the mathematics actually used by social scientists in their publications, the committee was able to compile a minimum list of the topics of which a general student of social science would need to have a clear understanding if he were to work with maximum effectiveness in his field. The list included:

(1) *Logarithms.* Common logarithms; numerical computation; compound interest and annuities; Napierian logarithms.

(2) *Graphs* (as a tool in the study of tabulated data). Plotting on ordinary and logarithmic papers; measurement of slopes and areas; graphical determination of maxima and minima; mean values; representation of three-variable relationships by means of three-dimensional diagrams and contours.

(3) *Interpolation.* By reading ordinary or logarithmic graphs; by proportional parts; by successive differences.

(4) *Equations and forms of curves.* Basic mathematical functions, their graphs, and the characteristic properties of the variations represented straight lines; parabolic and hyperbolic curves; exponential and logarithmic curves; logistic curves; sine and cosine curves.

(5) *Probability.* Combinations, binomial theorem, elementary probability; the normal probability curve—its form, table, and equation; probability and frequency distributions; probability and time series.

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[1] EDITOR'S NOTE—In 1930 the Social Science Research Council appointed a committee, with H. R. Tolley as chairman, to report on Collegiate Mathematics Needed in the Social Sciences. The report was presented in 1932, first to the Social Science Research Council, and then to the Mathematical Association of America, and appears in ECONOMETRICA, Vol. 1, April, 1933, pp. 197–204. Professor Griffin served as a member of that committee, in addition to spending six months in 1931 under the auspices of the Social Science Research Council interviewing European economists and biologists who had been using mathematics in their writings. Other material in ECONOMETRICA related to the use of mathematics in the social sciences, particularly economics, includes: Ragnar Frisch, "Editorial," Vol. 1, January, 1933, pp. 1–4; Joseph Schumpeter, "The Common Sense of Econometrics, Vol. 1, January, 1933, pp. 5–12; Irving Fisher, "Mathematical Method in the Social Sciences, Vol. 9, July–October, 1941, pp. 185–197; and the two Letters to the Editor cited in the footnote on page 205 of this symposium.

(6) *Elements of differential and integral calculus.* The significance of a derivative as a limit, as a rate or slope, as a frequency, etc.; the differential as an approximate increment; formulas for differentiating elementary mathematical functions, as u^n, $\log u$, $\sin u$, $\cos u$, uv, u/v; partial differentiations; procedure for determining maximum and minimum values; integration as the reverse of differentiation; relation between integration and summation; multiple integration.

(7) *Curve fitting* (mathematical principles). Plotting tabulated values, their logarithms, reciprocals, etc., to ascertain whether the tabulated points lie on a curve of simple form; determination of the curve through such points, or selected points within the scatter-diagram or chart; the method of least squares or other methods for obtaining the constants of curves of best fit.

The committee emphasized that illustrative material from the social sciences should be used freely, and the concepts and processes should be presented in such a manner as to make clear their application in the social sciences and provide a basis for recognizing their applicability or nonapplicability in particular problems. The report went into some detail on this phase of the course. It also urged that special effort should be made to acquaint the student with mathematical expression and reasoning as a mode of thought; and to cultivate a critical attitude in scrutinizing assumptions and the uses made of assumptions.

The committee deemed a course embracing such a list of topics to be not only a generally needed equipment for the effective undergraduate study of social science but also a desirable forerunner for courses in statistics which, being freed from the numerous excursions into purely mathematical preliminaries now commonly necessary, could devote themselves to a more mature treatment of statistics proper.

With respect to the training of many professional social scientists, especially in some branches of economics, the committee recognized that much mathematical knowledge was needed beyond the recommended general introductory course in mathematics and the following courses in statistics. It mentioned explicitly further calculus and the theory of probability, but failed to note the coming importance of familiarity with matrix or vector algebra and with such more advanced ideas as appear in recent works on cybernetics and the theory of games. For many professional social scientists, however, the more elementary courses would provide adequate orientation and equipment. This would probably be true, the committee believed, for most historians and for many sociologists, political scientists, and psychologists. Is it not true that many specialists in such fields are little more than laymen in theoretical economics? (I suspect that the techniques of econometrics are, and must continue to be, outside the range of competence of most social scientists.) For such laymen in parts of social science, as well as for the many students who take only a few undergraduate courses, the committee's

recommended courses still appear to be reasonably adequate if supplemented by very elementary matrix theory. I would, however, include a brief discussion of alternative postulational systems which give rise to rival systems of doctrine, with convenient illustrations in the non-Euclidean geometrics. Such a discussion is an eye-opener for virtually all students and is a powerful antidote for dogmatic tendencies.

The 1932 report concluded with two general suggestions: (1) that further study of the problem with particular reference to the specific needs of the different social sciences and the coordination between courses in mathematics and the different social science disciplines is highly desirable; and (2) that interested institutions be encouraged to proceed with the development of courses in mathematics designed primarily to meet the needs of students in the social sciences. The holding of this Symposium is a step toward carrying out the first of these suggestions, and it is highly desirable that something can be done about the second.

As to the latter, however, it is not easy to be optimistic. With few exceptions mathematicians are notably conservative with respect to the organization of their courses. Thus skepticism as to the practicability of treating so broad a range of topics in an introductory course of six to nine semester hours, and apprehension lest students who took such an introductory course should subsequently find themselves inadequately prepared for higher courses, have continued in spite of the notable success which at least one college has experienced with such a course for the past 38 years. Success will come only from a persistent well-organized campaign by social scientists to have courses set up which will be designed to meet the mathematical needs of their students, and, among other things, in the words of the Social Science Research Council president who appointed the 1930 committee, "to make it possible for social science students to get the little calculus they do need, without a frightful lot of calculus they do not need." The effort should be made, energetically!

Suggested Mathematical Curricula, GEORGE M. KUZNETS, University of California, Berkeley.

THE NEED for mathematical training of students in the social sciences arises from both theory and statistics. However, with the exception of economics (psychology a poor second) the lack of mathematical preparation is at present felt most keenly in connection with statistics rather than theory and it is this need which must be met in the immediate future.

Accordingly, it would seem to me that the undergraduate curriculum of students in the social sciences should include as a minimum the following courses: (a) a two-semester survey course dealing mainly with the elements of calculus, sprinkled liberally with examples and problems

taken from statistics and the social sciences; (b) a one-semester course in elementary probability. These courses should be taken before elementary statistics and theory (at least in economics); the latter offerings should be revised to take into account the background provided in (a) and (b).

The mathematical training of those intending to do serious work in the social sciences should go very considerably beyond the courses indicated above. It is now becoming possible to acquire a fair background in function theory at the undergraduate level. (At Berkeley three semesters of function theory and two semesters dealing with methods of attacking mathematical problems are offered to juniors and seniors.) The possibility of training in serious mathematics, without a heavy initial investment of time and effort in what might be called technical or manipulative mathematics, is of great significance to students in social sciences. I feel that it is precisely this kind of mathematics that is needed for a development of useful theory in the social sciences and a more skillful use and appreciation of statistical tools of analysis.

Mathematical Training Necessary for the Research Worker in the Social Sciences, HAROLD GULLIKSEN, Educational Testing Service.

As a minimum program for the mathematical training of the research worker in the social sciences, I would suggest:

(1) *Basic training in mathematics through differential equations.* It may be that some of the topics usually included in differential equations or among the preparatory courses could be eliminated. A panel of mathematicians and mathematically trained social scientists would be needed to determine the exact topics which might be eliminated in shortening this basic mathematical training.

(2) *Special training in the development and function of mathematical models.* This would be both a survey of models already developed and drill in the development of new models by students. The course should begin with well-established mathematical models in the physical sciences, such as Newton's laws of motion, field theories in electricity and magnetism, and possibly some theorems in mathematical biophysics. This material would help the student to obtain an appropriate understanding of mathematical models. The course could continue with mathematical models in the social sciences, such as have been developed in the psychology of learning, the field of social behavior (N. Rashevsky, J. Q. Stewart, G. K. Zipf), and epidemic theory (Lowell Reed). A one-year course with a prerequisite of differential equations would probably be the minimum that would be useful here.

(3) *Statistics.* One or two years including correlation theory and elementary analysis of variance. This work should emphasize the testing of the agreement between the mathematical model and the data. That is

to say, it should emphasize the finding of nonsignificant rather than significant differences.

(4) *The theory and logic of measurement.* This would be a one- or two-year course concerned with the basic logic of measurement, including modern developments in scaling methods, psychophysics, and probably work in test theory.

(5) *Special courses* should be arranged in each social science to give the students additional mathematics, statistics, or applications particularly relevant for his own field of specialization.

Since general recommendations coming from the National Research Council or the Social Science Research Council would probably secure more attention from university administrators, I feel that we should seriously consider the desirability of recommending to one or both of those organizations that they consider and report in a very general way on the topic of mathematical training appropriate for social sciences. Such a general recommendation should then be followed by concrete recommendations made by a joint professional group for the different special fields.

Needed Steps in the Mathematical Training of Social Scientists, HAROLD HOTELLING, University of North Carolina.

THE mathematics used in economics has grown rapidly to include not only elementary calculus but also differential equations, difference equations, calculus of variations, higher branches of geometry, and matrix algebra. To an even greater extent economics, and the other social sciences as well, use statistics. Efficient sampling and full utilization of available data call for modern statistical methods whose full understanding and satisfactory development require the use of some really advanced mathematical tools, and may shortly involve every known branch of mathematics.

The mere user of the results of social science, or ordinary citizen, may to some extent take on faith the results obtained with the help of mathematics he cannot understand, though it is notoriously harder to get him to accept expert opinion in the realm of economics, in which every man tends to fancy himself proficient, than, say, in physics or bridge design. Likewise some economists and other social scientists may take on faith the statistical methods and the contributions to such subjects as the incidence of taxation and the optimum rate of exploitation of exhaustible resources made by mathematics they cannot understand. But it is more likely that social scientists unacquainted with the necessary mathematics and therefore unable to follow the derivations will either ignore or misunderstand the conclusions, and will misuse statistical methods.

The social sciences will not be in a satisfactory state until the specialists in them know a good deal more mathematics than at present. This

will be possible for any considerable numbers in coming generations only if fairly drastic reforms are first made in teaching of mathematics. One such reform is the introduction into mathematical textbooks of a few more examples drawn from the social sciences, such as those involving minimum cost, maximum profit, and consumers' surplus in elementary calculus, and the application of the calculus of variations to maximizing discounted profit over a period of time, as in Fite's *Advanced Calculus.*

The principal reform needed, however, is the pushing of the mathematical program down into lower levels in the schools, together with improvements in the mathematical knowledge of the teachers. The practicability of this is amply demonstrated by experience in numerous European countries, where students entering the university at the age of eighteen have had a substantial course in calculus. Moreover, many interesting topics usually reserved for graduate students specializing in mathematics can be put into forms in which they will be received enthusiastically by quite young children, as was illustrated by Professor Edward Kasner's success in teaching differential geometry in a New York kindergarten. A young economist should have his differential equations behind him, not ahead of him, when he is trying to get a grip on the foundations of economics.

To bring about the needed reform in the teaching of mathematics will require teachers and textbook authors who know more of mathematics and its complications. The change in mathematical accomplishments of teachers will in practice be possible only with a change in licensing requirements to stress mathematics, even though this means sacrificing some of the repetitious courses in "education" now required of prospective teachers.

Suggested First Steps in Mathematical Economics, WILLIAM JAFFÉ. Northwestern University.

ONE of the reasons for the relatively slow progress made thus far in rendering the application of mathematics to economics more attractive in academic curricula is our failure to develop an appropriate *introduction* to the study. Most discussions dealing with the question concern themselves with the items that should be included in courses in mathematics for economists or in courses on mathematical economics. We hear an infinite series of urgent recommendations of what students "ought to know." Where the group of discussants is large enough, voices are raised as at an auction sale, each out-bidding the other in the esoteric refinement of his specific offering. The bidders are too often those whose interest lies principally in the heuristic function of the mathematical method. From their special point of view, the advice of research workers is valuable. But we must not lose sight of another function of the method, and that is its didactic function. However reluctant we may be to descend

to this lower order of consideration, we must acknowledge that to neglect it altogether is to defeat our purpose: the mathematical training of social scientists.

When I speak of an introduction, I am not using the term as Bowley does, when apparently without facetious intent he calls his *Mathematical Groundwork of Economics* "an introductory treatise." Having come rather late myself to a realization of the import of mathematics in relation to economics, I was inveigled by Bowley's sub-title into seizing upon his book to start me off, only to throw up my hands in utter despair at the first page. I then discovered such helpful devices as Irving Fisher's *Infinitesimal Calculus*, Leseine and Suret's *Introduction Mathématique à l'Etude de l'Economie Politique*, and Griffin's *Introduction to Mathematical Analysis* (this was before R. G. D. Allen's book appeared); and I got a modest start.

Further study and teaching have led me to the conviction that this way of introducing one's self or anyone else to the subject is not the best way. Most students, whether on the graduate or undergraduate level, experience great difficulty, even after they have run through the whole usual gamut of advanced mathematical courses, in seeing how their knowledge of mathematics can be usefully applied to economics. I doubt whether I am alone in observing that often students less well equipped with technical mathematics show a greater intuitive insight into problems involved. That is why I am not too sanguine about proposals that confine themselves simply to favoring this or that preliminary set of mathematics courses as prerequisites for work in the fields we have in mind.

In my opinion, a consideration of the architectural coherence of courses (and textbooks) as a whole should have priority over considerations of specific content. Without careful attention to the principle of architectonic unity modern courses and treatises either degenerate into an amorphous agglomeration of unconnected topical developments or, without being formless, are given a structure dictated by the traditional requirements of mathematical instruction instead of conforming to the nature of the social science in which we are primarily interested. Such an alien structure is found, for example, in Allen's *Mathematical Analysis for Economists*, and in Crum and Schumpeter's *Rudimentary Mathematics for Economists and Statisticians*. Both follow a mathematical, and not an economic, order of development.

What would be far more useful as well as far more effective in enlisting a wider interest in a rigorously mathematical conception of economics and in starting future economists on their career is a text, say at the intermediate level, which is economic in design and only incidentally mathematical, rather than the other way round. In such a text it is per-

fectly possible to present and elucidate, as it moves along, progressively more complex and sharper mathematical tools. Obviously patient pedagogical experimentation must precede the composition of an introductory textbook embodying this plan. I have found such experiments rewarding, for I have observed students, who were otherwise disinclined to engage in the abstract manipulation of mathematical tools in the rarified atmosphere generated by pure mathematics, become entranced when they saw these tools in actual use in the economist's workshop. Moreover, having once gained an insight into the general meaning of mathematical concepts via economics, they acquired a taste for the cultivation of mathematics for its own sake.

Model Building and Statistical Inference, THEODORE W. ANDERSON, JR., Columbia University.

THE FOLLOWING remarks are directed toward the question of the training of social scientists for doing research in their respective fields with the aid of mathematical tools. Broadly speaking, mathematics is used in the social sciences in two ways: (1) in stating and developing the theory in a particular field and (2) in dealing with statistical problems.

By formulating a theory in mathematical terms one can make his definitions and state his assumptions unambiguously. The more fruitful result of the mathematical formulation is the derivation of the logical consequences of the assumptions by use of mathematical theorems.

Once the theory has been set up in mathematical form, there is the problem of relating this theory to the observed facts. This may involve testing the validity of the assumptions or conclusions, or it may involve the estimation of elements of the theory which are otherwise unspecified. There are usually discrepancies between the theoretical model and the observations. In each case the model can be reformulated in probability terms so that the elaborated model includes an explanation of the discrepancies. The proper methods of statistical inference can then be deduced from consideration of this probability model and the available data.

As an example of these stages in social science research we might consider the study of the economic structure of the United States. Extension of classical mathematical economics leads to the statement of the macro-economic model in terms of a number of equations holding between the relevant quantities. These complicated equations are then approximated by a set of linear equations. To account for the discrepancies between this model and the observed data the linear equations are made random. Then one uses statistical tools for testing hypotheses about the system of random equations and estimating the coefficients of the equations and other parameters.

Examples of model-building and subsequent statistical inference can be given from other fields. There are the analysis of mental abilities and the related statistics of factor analysis, the scaling of attitudes, "factor analysis" of attitude data, and analysis of changes in attitude over time.

In mathematical training of social scientists for the purpose of conducting research, I think the principles of mathematics should be emphasized. It is more important to understand the concepts, such as a derivative or a linear transformation, than to be able to get solutions to numerical problems. In building theoretical models one draws upon the calculus, the theory of functions of real variables, calculus of variations, and differential equations. When one simplifies one of these models to linear models one also needs vector or matrix algebra. In developing these models in probability terms and in relating these to actual data, one needs to know probability and statistics. In many cases other parts of mathematics are needed—in particular, mathematical logic and the newly developed theory of games.

Mathematics for Theoretical Economists, GRIFFITH C. EVANS, University of California, Berkeley.

STUDENTS who are interested in theoretical economics are not necessarily of one kind. The interest of one may be mainly historical or critical; it may be restricted to a single period. In that period it may center on the relation between economics and a phase of literature or history or the interchange of ideas between philosophy and economics. It may focus on one philosopher and one economist. An economist draws on whatever he has found fruitful in literature, history, philosophy, or mathematics. He is not discouraged at not knowing everything.

A respectable portion of economics, for a century and a half, at least, has been formulated with an eye on mathematics. Indeed, several chapters, like the theory of interest and annuities and the actuarial basis of insurance, have been detached from the main book and made into separate volumes. Both are primarily economic in orientation, and it is mere accident that one is sterile mathematically while the other is rich in mathematical problems. Beyond such more or less distinct studies, however, we continue to extend our gaze over other parts of the field, with the notion that since Isaac Newton did so much for others, he must have had his hand ready also for a helping grasp of ours. And yet there remain parts of the field which seem obviously to be the kind of soil which mathematics would fertilize, but are still uncultivated. The theory of accounting should call for something more in its development than it now recieves.

The reason why so much of the main body of economic theory has not succumbed to a mathematical treatment is not difficult to imagine—at

least a possible reason. Economics must make its theories intelligible to business men and functionaries. We hope that the man in the street and the statesman in responsible office both may be informed and guided in the insight gained from its study. Thus theory tends to grow in directions which are not overgrown with technical flora, the practical man looking askance at paths so obscure to him that he cannot force his way through them with common sense and experience. As a portion of the public ourselves we must hope that practical judgment in matters of state will not degenerate into mere reliance on experts, but will continue to be based on informed intuition and imagination.

Thus it seems reasonable to expect that there will always be a broad range of economics which can be explained and interpreted without the help of the methods of partial differential equations, Hilbert spaces, and advanced topology, and that there will remain studies which can be performed adequately by persons who have not been trained in those mathematical disciplines. Much of mathematics is a sort of cumulative arithmetic, with an indefinite extension, as in some geometrical aspects, to limiting situations; and therefore much of economic theory can be made clear by arithmetical and geometric device and illustration. On the other hand, such exposition, not done with surpassing skill, can become surpassingly dreary and wearisome. It is often repetitious and uninformed.

Certainly any expositor will be wise if he has more tools at his command, both in economics and mathematics, than those of which his average reader is conscious. He can make ideas clearer in elementary terms if he has resources, for the obtaining and analyzing of them, which save time and effort and exhibit clearly to his own mind their essential nature reduced to abstract connections. He is best off if he can make himself a master of two styles, one for general exposition as a servant of the public, and one for communication to other experts. As much of the abstract mathematical technique as he can command will be valuable for him, because, however useful, a historical sense is not a complete substitute for logic.

It is natural to say that an economist should know something of calculus, matrices, even differential equations, and a fair amount of technical mathematical statistics—the last in order that he may evaluate economic data toward the analysis of those situations where business men, politicians, and statesmen have carried on large scale experiments. But for most, I think that we can be satisfied with much less, namely a good knowledge and *habitual use* of algebra and geometry.

For anything further a person should study mathematics for the same reason that he studies economics, to wit, that he likes the subject. Moreover, in each subject he should study what he pleases, and he should be both a mathematician and an economist. For there is nothing that we

can prophecy will not be turned to account. Liberal minds are the prize ornaments and possessions of society, and no restraint and little direction should be imposed upon their development. There will always be mathematical statisticians who will be interested in economic data and economists who are interested in the philosophy and technique of probability. Part of their natural ambition is to make the results of their study available for others. Likewise there will always be mathematicians who will find out how the social and business world looks to them, in the light of their particular attainments, and economists who wander far afield in the exploration of new mathematical ideas.

Mathematical Training of Undergraduates in Agricultural Economics, R. L. Anderson, North Carolina State College.

The problem of requiring agriculture economics undergraduates to take mathematics through the calculus is difficult for these reasons: (1) Their precollege mathematical training is often quite inferior. (2) There are few elective courses because of the large number of required courses in technical agriculture and natural science. This has resulted in the present watered-down agricultural mathematics courses. (3) There is a cynical attitude towards economic theory courses at many agricultural colleges.

A combined course in mathematical economics and mathematics through the calculus was introduced at Iowa State College by Gerhard Tintner, and later I taught it at North Carolina State College. Since some agricultural schools may be able to include such a combined course in their curricula, I will mention the topics covered: (1) Linear demand and supply curves with graphical and algebraic solutions. (2) Use of quadratic equations, logarithms, and exponentials. (3) Geometric progressions. (4) Finite differences. (5) Use of differential calculus for marginal analysis, elasticity, and point of maximum profit. (6) Utility theory. (7) Use of integral calculus for calculations over time. (8) Elementary statistics through moments, regression, and correlation.

On the basis of my experience in teaching both the combined mathematics-economics course and some undergraduate mathematics courses, I recommend the following courses in mathematics for undergraduates in agricultural economics: (1) A strong freshman mathematics course through analytic geometry, if possible. The students should understand that this course will be required for the mathematical economics which will follow. The course should be taught by someone familiar with the social sciences so that he can use social science examples. (2) Either a course in calculus and one in mathematical economics or a combined course such as that used at Iowa State College. If the combined course is taught, it should be fitted into the second-year program before the students have forgotten their freshman mathematics. I believe that two

years of mathematics is preferable, but most agricultural curricula are too full to permit this. These courses should be coupled with some required courses in economic theory. I do not believe that the combined mathematics-economics course should be taught if it is not required of all agricultural economics majors.

Remarks on the Training of Social Scientists, MARTIN BRONFENBRENNER, University of Wisconsin.

IF TRAINING is to equip individuals who are not basically mathematicians with sufficient technical competence to read some appreciable part of the current technical literature in their own fields, such training can very well take the form of "bits and pieces" extracted from the body of mathematics and arranged in special courses for future social scientists. The particular "bits and pieces" involved will presumably be chosen on the basis of application in current social science literature. On the other hand, when training is to provide a background for original contributions to the mathematical and statistical literature of the field, it is probably administered more economically in the long run by the departments of mathematics, physics, and perhaps astronomy.

What should the student be recommended to sacrifice in order to obtain the additional mathematical training which we urge upon him? Here the point should be stressed which was expressed previously in my reply to "Pertinax,"[1] namely, that his load of undergraduate social science courses should be reduced rather than his general training in languages, mathematics, history, philosophy, etc.

[1] See Letter to the Editor by Pertinax in "Criticism Invited," ECONOMETRICA, Vol. 17, January, 1949, pp. 90–92, and the reply by M. Bronfenbrenner, ECONOMETRICA, Vol. 17, July–October, 1949, pp. 251–252.

NOTICE OF CHANGE IN WEST COAST MEETING

August 1–5, 1950

The regional meeting of the Econometric Society on the West Coast will be held on the Berkeley campus of the University of California starting Tuesday afternoon, August 1, and continuing until Saturday noon, August 5, instead of in Los Angeles in June as announced earlier. The meeting will be concurrent with the Second Berkeley Symposium on Statistics and Probability and with a meeting of the Institute of Mathematical Statistics, and some joint sessions are being planned in addition to sessions for contributed papers on economic theory.

Abstracts of papers intended for presentation at the Berkeley meeting and other correspondence regarding the meeting should be addressed to the program chairman, Professor Kenneth J. Arrow, Department of Economics, Stanford University, Stanford, California. Local arrangements will be handled by Professor George M. Kuznets, University of California, Berkeley. The complete program committee is listed in the January, 1950, issue of ECONOMETRICA.

Further details as to the program and arrangements will be contained in a preliminary program to be mailed American members in advance of the meeting.

ANNOUNCEMENT OF HARVARD MEETING

August 31–September 6, 1950

The Econometric Society will hold a meeting at Harvard University, Cambridge, Massachusetts, Thursday, August 31, to Wednesday, September 6, 1950. The Harvard meeting will take the place of the summer meeting which the Society customarily holds in conjunction with the mathematical associations, and will tend to stress papers of a more technical nature than those presented at the December meetings with the social science organizations. The time and place of the meeting coincide with the International Congress of Mathematicians, but the two meetings are simultaneous rather than joint, though some cooperative arrangements will be made.

The program committee consists of Arthur Smithies, Harvard University (chairman); R. G. D. Allen, London School of Economics; James Duesenberry, Harvard University; Thornton C. Fry, Bell Telephone Laboratories; Leonid Hurwicz, University of Illinois; Herbert Marshall, Dominion Bureau of Statistics, Canada; Jacob Marschak, Cowles Commission for Research in Economics; Guy Orcutt, Harvard University; Paul Samuelson, Massachusetts Institute of Technology; William B. Simpson, Cowles Commission for Research in Economics (ex officio); Gerhard Tintner, Iowa State College, Albert W. Tucker, Princeton University; and Frederick V. Waugh, Council of Economic Advisors.

Abstracts of papers intended for presentation at the Harvard meeting should be addressed to Professor Arthur Smithies, Department of Economics, Harvard University, Cambridge 38, Massachusetts, prior to June 1. Papers accepted by that date will be listed in the announcement to be mailed to American and Canadian members in early June.

ANNOUNCEMENT OF VARESE MEETING

September, 1950

The twelfth European meeting of the Econometric Society will be held at Varese, Italy, this year, about the 18th, 19th, and 20th of September. The exact dates are still tentative since it is desired to prevent conflict with the projected meeting of the International Economic Association at Grenoble, in France.

The following have been approached to serve as program committee for the Varese meeting: Dr. Costantino Bresciani-Turroni, Banco di Roma, Via Bocchetto 4, Milano, Italy, (chairman); Professor R. G. D. Allen, London School of Economics; Professor Eraldo Fossati, Università Degli Studi di Trieste; Professor Ragnar Frisch, University of Oslo; and Professor René Roy, Paris, France. Professor Fossati has accepted the responsibility for local arrangements for the meeting.

Members interested in participating in the Varese meeting are encouraged to contact the program committee for further information.

ANNOUNCEMENT OF CHICAGO MEETING

December 27–30, 1950

The American winter meeting of the Econometric Society will be held in Chicago, Illinois, Wednesday, December 27, to Saturday, December 30, 1950. The program will include sessions for invited and contributed papers as well as some joint sessions with the American Economic Association, the American Statistical Association, the Institute of Mathematical Statistics, and related organizations.

Clifford Hildreth, Cowles Commission for Research in Economics, will serve as chairman of a program committee including Harold Barger, National Bureau of Economic Research and Columbia University; Howard R. Bowen, University of Illinois; Harold T. Davis, Northwestern University; G. A. Elliott, University of Toronto; Merrill M. Flood, The RAND Corporation; Francis McIntyre, California Texas Oil Company; Oskar Morgenstern, Princeton University; J. J. Polak, International Monetary Fund; Ross Robertson, University of Tennessee; Charles F. Roos, The Econometric Institute, Inc.; William B. Simpson, Cowles Commission for Research in Economics (ex officio); and Holbrook Working, Stanford University.

Abstracts of papers to be considered for the December meeting and other correspondence relative to the meeting should be addressed to Professor Clifford Hildreth, ℅ The Econometric Society, The University of Chicago, Chicago 37, Illinois. Further details will be announced as plans progress.

CARE BOOK PROGRAM

The Econometric Society has received an appeal from CARE for aid in its Book Program for education and cultural rehabilitation in Europe and Asia. At the request of UNESCO and other groups concerned with world reconstruction and exchange of information, and with their cooperation, CARE has provided facilities for sending the latest technical and professional information to overseas libraries, universities, and research groups. The most urgent need is for books dealing with recent research and the development of scientific techniques. Contributions and requests for further information may be sent to CARE Book Program, 20 Broad Street, New York 5, New York.

Eugen Slutsky

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THE WORK OF EUGEN SLUTSKY

BY R. G. D. ALLEN

EUGEN SLUTSKY, whose death has recently been reported, was an outstanding mathematician and statistician. His work has had a great and lasting influence on the development of econometrics in two important fields, the theory of consumer's behaviour and the analysis of time series. In each case his basic article remained almost unknown for many years, was later rediscovered, and had an increasing effect in shaping the work of others in the field.

During the 1920's and early 1930's, Slutsky published a number of statistical articles on stochastic processes and time series analysis, some in French, Italian, and German. His main article [23], now a classic contribution, appeared in Russian, with only a brief summary in English, in a publication of the Moscow Conjuncture Institute in 1927. His treatment of the time series problem was akin to that of Yule in the famous articles of 1921 and 1926 in the *Journal of the Royal Statistical Society*[1] and together they had a very great influence on the later work in the field. Slutsky's contribution, however, remained little known for some years, until Henry Schultz, whose eagle eye missed very little in his field, was responsible for the preparation of a translation. An English version of the work, with the addition of later material, was finally prepared by Slutsky and published in this Journal in 1937 [42].

The main object of this work of Slutsky's was to demonstrate that an oscillatory series could be generated from a random series by taking a moving sum or difference, with or without weights and with or without repetition of the process. The oscillatory series so generated displayed

[1] "On the Time Correlation Problem with Especial Reference to the Variate-Difference Correlation Method," *Journal of the Royal Statistical Society*, Vol 84, July, 1921, pp. 497–526, and "Why Do We Sometimes Get Nonsense-Correlations between Time-Series? A Study in Sampling and the Nature of Time-Series," *Journal of the Royal Statistical Society*, Vol. 89, January, 1926, pp. 1–64.

EDITOR'S NOTE: The following information regarding Eugen Slutsky (Evgeniĭ Evgenievič Sluckiĭ) is drawn from memorial articles written by A. N. Kolmogorov [*Uspekhi Matematicheskikh Nauk*, Vol. 3, No. 4 (26), 1948, pp. 143–151] and N. Smirnov [*Izvestiia Academii Nauk S.S.S.R.*, Mathematical Series, Vol. 12, 1948, pp. 417–420].

Eugen Slutsky was born in 1880. He studied in the department of physics and mathematics at the University of Kiev. In 1901 he was expelled from the university and conscripted into the army, together with other students, because of participa-

approximate regularity, with varying length and amplitude of oscillation, and they were very similar to many economic time series. Under certain circumstances, the generated series could be made to approximate very closely to a sine-curve. Slutsky's results have been of great value in research into the problem of whether a moving-average trend distorts the true oscillations in a series and into the wider question of the structure of economic time series. Because of the mathematical complexities involved, the present approach to time series analysis, by Kendall, Orcutt, and others, is still the same as Slutsky's, a combination of deduction with experimentation on actual or (more usually) constructed series.

Slutsky's basic article in the theory of consumer's behaviour was earlier; it appeared under the title "Sulla teoria del bilancio del consumatore" in *Giornali degli Economisti*, 1915 [6]. He put his argument in a highly mathematical form, without much elaboration of the economic significance of his results, and he was unfortunate in that he published in wartime. Consequently, like another mathematical article on the same topic by Johnson,[2] his work received little notice at the time. It was only rediscovered in the middle 1930's by those then developing the theory and measurement of consumer's demand. Even Henry Schultz did not discover Slutsky's article until around 1934, and he included an account of it in his "Interrelations of Demand, Price and Income"[3] in 1935. Independently, though a little later, Hicks and I were led back to Slutsky's original work by various references to it, and I published a summary of it in the *Review of Economic Studies* in 1936.[4]

It can be said, without doubt, that the present theory of consumer's behaviour, as developed by Hicks and others, is essentially as much a

tion in student revolts, but was returned to the university as a result of further student protests in the country. In 1902, however, he was again expelled and deprived of the right to study in any Russian institution of higher learning. From 1902 to 1905 he studied in the engineering department of the Institute of Technology at Munich, Germany. In 1905 he received permission to study in Russia and entered the department of law, University of Kiev, with the intention of applying mathematical methods to economics. He graduated from the university with a gold medal in 1911. He became a member of the faculty at Kiev Institute of Commerce in 1913, reaching the position of full professor in 1920. Three years earlier he received a degree in political economy from the University of Moscow. From 1926 on, he was a staff member of the Central Statistical Board in Moscow. In 1934 he became a staff member of the Mathematical Institute of the University of Moscow, and in 1938 became a member of the Mathematical Institute of the Academy of Sciences of the U.S.S.R. Slutsky died March 10, 1948, in Moscow.

[2] "The Pure Theory of Utility Curves," *Economic Journal*, Vol. 23, December, 1913, pp. 483–518.

[3] *Journal of Political Economy*, Vol. 43, August, 1935, pp. 433–481.

[4] "Professor Slutsky's Theory of Consumers' Choice," *Review of Economic Studies*, Vol. 3, February, 1936, pp. 120–129.

development of Slutsky's work as of Pareto's. Quantities demanded by an individual consumer as functions of given prices and income are given by the conditions for an extreme value of an ordinal utility function. The stability conditions, for a maximum as opposed to a minimum, then serve to determine the properties of the demand functions. i.e., the variation of demand with changing prices and income. A good deal of confusion of thought has arisen because of the fact that the restraint set by the condition of given total expenditure is linear and homogeneous in the variables (prices and income). A proportionate increase in all prices with income fixed is equivalent to a decrease in income with all prices fixed, each corresponding to a decrease in real income, i.e., a shift to a lower level on the consumer's preference scale. Slutsky's achievement was to show that any change in prices and income must be analysed into two parts. The first is a change in relative prices with fixed real (not money) income. This is the substitution effect and the consumer maintains a given indifference level. The second part is the balance of the price change (a proportionate shift in all prices), which can be translated into an equivalent change in income and added to whatever change there may be in money income to give the variation in real income. This is the income effect and the consumer shifts from one indifference level to another. The two effects turn out to be independent and additive, as indeed is intuitively clear.

There are two equivalent ways of attacking the problem. For the substitution effect, real (not money) income is kept fixed *either* by taking the utility level constant and minimising expenditure for each set of market prices *or* by adding a compensating change in money income to given price changes in order to maintain the original indifference level. In the analysis of the income effect, *either* the minimised expenditure can be compared with actual money income *or* the actual change in money income can be adjusted for the compensating income change. The second was the method adopted by Slutsky, and it is certainly the easier to develop and expound.

With Hicks's notation,[5] n commodities are demanded in amounts x_r at prices p_r $(r = 1, 2, \cdots, n)$ and with income M. A given direction of price change is denoted by the differentials dp_s, and the compensating income change to maintain the indifference level is

$$dM = \sum_{s=1}^{n} x_s\, dp_s.$$

The change in demand for the rth commodity is

$$dx_r = \sum_{s=1}^{n} \frac{\partial x_r}{\partial p_s} dp_s + \frac{\partial x_r}{\partial M} dM,$$

[5] J. R. Hicks, *Value and Capital*, second edition, Oxford: Clarendon Press, 1946.

i.e.,

$$dx_r = \sum_{s=1}^{n} \left(\frac{\partial x_r}{\partial p_s} + x_s \frac{\partial x_r}{\partial p_s} \right) dp_s .$$

This is the variation in demand for a compensated price change, and the expressions $[(\partial x_r/\partial p_s) + x_s(\partial x_r/\partial p_s)]$ represent the substitution effect. These expressions were first defined by Slutsky and analysed by him in terms of the utility functions. He termed them residual variations in demand for a compensated variation in price.

As Slutsky showed, the substitution expressions (residual variations) are limited in various ways by the conditions for equilibrium and for stability. What is now seen, after much development of the basic Slutsky theory, is that the limitations boil down to one equation and one inequality:

$$(1) \qquad \sum_{r=1}^{n} p_r\, dx_r = 0; \qquad \sum_{r=1}^{n} dp_r\, dx_r < 0.$$

The first (derived from equilibrium conditions) expresses the fact that, if there are compensated price changes, then some demands rise and others fall. The second is one consequence of the stability conditions. If only one price (p_r) changes (say, increases), then $dx_r < 0$, i.e., a single compensated price increase results in a reduced demand for the commodity. The demands for other commodities may rise or fall, but at least one must rise. If more than one price changes, nothing definite can be said about the direction of change of any particular demand. All inequality (1) says is that, very broadly, demand falls tend to be associated with prices which have risen relatively to others.

Slutsky himself did not reach this conclusion; it is the development of a theory originated by Slutsky. As so often happens, the line to advance in the end appears to have been a spiral. For we find we can say very little in general about the variation of demand, and what we can say is simple. Moreover, the results turn out to be derivable directly, without the mathematical scaffolding erected so painfully, and they can be extended to apply to any price changes and finite differences and not only differentials. Take any two situations on the same indifference level (i.e., a compensated price change), the first with purchases x_r at prices p_r, the second with purchases $(x_r + \Delta x_r)$ at prices $(p_r + \Delta p_r)$. The second purchases must cost more than the first if both are reckoned at the first prices (since the first purchases are actually made at these prices), and similarly at the second prices:

$$\sum_{r=1}^{n} p_r(x_r + \Delta x_r) > \sum_{r=1}^{n} p_r x_r ;$$

$$\sum_{r=1}^{n} (p_r + \Delta p_r)x_r > \sum_{r=1}^{n} (p_r + \Delta p_r)(x_r + \Delta x_r),$$

i.e.,

$$\sum_{r=1}^{n} p_r \Delta x_r > 0; \qquad \sum_{r=1}^{n} \Delta p_r \Delta x_r < 0. \tag{2}$$

The limiting form of (2) is (1), and (2) is called by Hicks the generalised law of demand. The difficulty is to determine exactly what compensating change in money income is needed to maintain the given indifference level. If this change is ΔM, then all that can be said is: $\sum_{r=1}^{n} \Delta p_r(x_r + \Delta x_r) < \Delta M < \sum_{r=1}^{n} x_r \Delta p_r$, so $\Delta M = \sum_{r=1}^{n} x_r \Delta p_r$ only approximately and only if all changes are small.

Samuelson goes even further, in *Foundations of Economic Analysis*, 1947,[6] and suggests that the theory can be divorced entirely from indifference levels and expressed only in terms of "revealed preferences." His basic condition of consistent consumer behaviour is:

$$\text{If} \quad \sum_{r=1}^{n} p_r \Delta x_r \leqslant 0, \quad \text{then} \quad \sum_{r=1}^{n} (p_r + \Delta p_r)\Delta x_r < 0.$$

In particular:

$$\text{If} \quad \sum_{r=1}^{n} p_r \Delta x_r = 0, \quad \text{then} \quad \sum_{r=1}^{n} \Delta p_r \Delta x_r < 0. \tag{3}$$

In (3), which replaces (2), there is implied a compensating income change

$$\Delta M = \sum_{r=1}^{n} \Delta p_r \, (x_r + \Delta x_r) < \sum_{r=1}^{n} x_r \Delta p_r .$$

This compensation is not to maintain the consumer at a given indifference level (the indifference map having been discarded from the theory); it is to keep him on the original price plane, i.e., to keep him to purchases which could have been made with the original prices and income. The limiting form of (3) is again (1).

Slutsky wrote before his time and his was the contribution of a mathematician. But the concept of compensated price changes which he introduced has become so well established that it dominates the theory of consumer's behaviour. There is little to say about the income effect; it is generally positive, but can be negative, and it is possible to estimate its value from economic data. It is the substitution effect which interests the economist and puts the problems up to the econometrician.

It is unfortunate that, for so long before his death, Slutsky was

[6] Cambridge, Mass.: Harvard University Press.

almost inaccessible to economists and statisticians outside Russia. He opened up new areas but left them to be explored by others, and the exploration even now is far from complete. His assistance, or at least personal contacts with him, would have been invaluable.

London School of Economics

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EMPIRICAL DETERMINATION OF A MULTIDIMENSIONAL MARGINAL COST FUNCTION[1]

By A. R. Ferguson

An empirical cost function for fuel consumption in air transportation is obtained from a consideration of the engineering determinants of the relationship between physical inputs and physical outputs. The accuracy of fuel consumption estimates so obtained is considered in the light of operating data for Northeast Airlines. Marginal cost functions are derived which facilitate the analysis of the impact upon cost, not only of changes in output and factor prices, but also of changes in quality of the product and the techniques of production. The advantages and limitations of the approach are indicated.

INTRODUCTION

This paper employs an essentially new approach,[2] for economists, to the problem of the empirical determination of cost functions. The aim of the method is a more reliable, more general, more precise determination of costs than is possible with the various statistical methods frequently used in cost studies. The method employs the engineering determinants of physical inputs relative to physical outputs, essentially converting the engineering relationships into production functions.

This particular paper deals with fuel consumption in air transportation. It develops a cost function from which it is possible to ascertain the marginal fuel costs (in physical units) of variations in the quantity of output, in the quality of output, and in techniques of production. In principle the method is not unlike the one apparently employed by M. Louis Bréguet in a French study of air transport costs reported earlier in Econometrica [19, pp. 249–259]. The present study develops the determination of fuel consumption more fully than did Bréguet's exploratory work.

[1] The method employed in this paper was developed in a doctoral dissertation written with the invaluable help of Professors George P. Baker, Edward S. Mason, and, especially, Wassily W. Leontief, of Harvard University. The empirical research embodied in the second part was made possible by a grant from the Bureau of Population and Economic Research of the University of Virginia, directed by Dr. Lorin A. Thompson. I am indebted to Professors David McC. Wright and D. Rutledge Vining and to Mr. Howard Nicholson, of the University of Virginia, and to Professor Armen Alchian, of the University of California at Los Angeles, for having read the entire manuscript and for offering many valuable suggestions. Mr. Victor Baron of the University of Virginia has spent a great deal of time checking equations. Through his thoroughness many errors have been eliminated.

[2] The method is outlined in detail in the present writer's Ph.D. thesis [12]. A very similar method is developed independently in Hollis Chenery [4, 5]. Some elements of the method were employed by Raymond Bressler [3].

The basic physical determinants of fuel requirements are set down in equations which show the nature of the interrelationships in relatively precise quantitative terms and which are sufficiently general to take account of technological change and any changes in operating conditions. Part I is devoted to deriving these equations. Part II is essentially a digression. Data obtained from Northeast Airlines are employed showing that in one case the use of the equations yields a prediction which is reasonably accurate. It should be clear that this does not in any sense constitute empirical proof of the validity of the equations (indeed their formal validity rests upon the principles of thermodynamics and aerodynamics). The case presented merely shows that the equations are operational in the sense that the data required for their use are available to airline management, and hence, presumably, to regulatory authority, and that with them it is possible to obtain predictions of fuel consumption which are of the right order of magnitude. Part III elaborates the economic significance of the equations. It follows directly from equation (6) in Part I and is not dependent upon Part II.

An examination of equations (3), (6), and (7) and a more or less thorough reading of Part III may well serve the theorists' purposes. Nevertheless, Part I is indispensable to establish the validity and limitations of the third part; and with Part II it illustrates the amount and complexity of the detailed material which must be handled if one is to obtain reliable empirical statements of the quantitative relationships which are (often glibly) assumed to be available in the theory of the firm.

A complete analysis of airline costs would require a similar analysis of the determinants of all the input requirements coupled with a statement of the influence of factor price interrelationships upon the combination of the factors. The present study assumes the price of the factor in question, fuel, is constant, but changes in the price of any other factors, through their influence on the combination of the inputs, can be treated as variables.

So far as I know, nothing in the large literature on airline fuel consumption both deals with independent variables which are significant from the point of view of economics and treats the problem in a sufficiently general manner to permit handling all kinds of problems regardless of the type of aircraft employed and the kind of operation in question.[3] The scientific and engineering works take as variables entities which are not significant in the economics of airline operation, while treatments of the business and economic aspects deal with aircraft currently in operation, restrict their discussions to current prac-

[3] However, see [14, 19].

tice, or otherwise fail to take account of some factors which do vary between types of operations.

I. DERIVATION OF THE FUEL EQUATION

In Part I, as has already been mentioned, the quantitative determinants of fuel consumption will be set out algebraically.[4]

The power required by an airplane in level equilibrium flight depends upon the drag and lift characteristics of the airplane, the speed and altitude of flight, and the gross weight of the airplane at the moment in question. All these factors appear in the following equation [15, p. 421]:

$$P = \frac{(b_1 s_1 + s_2)\rho V^3}{2} + \frac{2W^2}{b_2 s_1 \rho V}, \tag{1}$$

where b_1 represents the drag coefficient for zero lift; s_1, wing area in ft.^{2}; s_2, the parasite area in ft.^{2}; b_2, a factor in the coefficient of induced drag, frequently in theoretical discussions taken as 3.1416 times the aspect ratio; P, power in ft-lbs/sec.; ρ, air density in slugs per ft.^{3}; V, velocity (True Air Speed) in ft/sec.; and W, gross weight in lbs.

The detailed derivation of the equation will not detain us; it is accepted as valid because of its source. However, a word of interpretation may facilitate the understanding by economists of its significance. The power required, P, is the power necessary to propel the airplane through the air. In equilibrium the power is required only to overcome the resistance of the air. This resistance is derived from two sources: first, the parasite drag, i.e., the frictional and frontal resistance which the air would present to a body of the size and shape of the airplane if its weight were supported other than by its dynamic reaction with the air. This force is represented in the first term on the right side of the equation. The second source of resistance is the induced drag, the resistance generated in the reaction of the air against the wing to overcome the downward pull of the weight of the airplane. This quantity is represented in the second term on the right.

The function of an engine is the conversion of the energy latent in its fuel into useful power. The power produced is dependent upon the combustion energy of the fuel, the rate at which the fuel is consumed, and the efficiency with which it is utilized. Thus [15, Chapter III, *passim*],

$$P = M_f \cdot c \cdot e_t \cdot e_p, \tag{2}$$

[4] In this effort I have enjoyed the cooperation of several prominent scientists in the field of aeronautics. I am particularly indebted to Professor Edward S. Taylor, of the Massachusetts Institute of Technology, who has given large amounts of his time in personal conversation answering my questions and pointing out ways in which to simplify my work, and who has made available to me reprints of some of his own writings and even some of his unpublished work. Dr. Phiroz J. Theodorides, visiting lecturer in aeronautical engineering at Harvard University, has also been very cooperative. He has called several important works to my attention and has taken much of his time in discussing my problems. Professor Richard von Mises of Harvard University has also answered several of my questions. Mr. Henry Maier, an assistant in the Department of Engineering at Harvard, read a preliminary manuscript, pointed out source material, and checked the accuracy of some of the work. It seems hardly necessary to add that none of these are in any way responsible for errors or omissions.

where P represents power in ft-lbs/sec.; M_f, fuel consumed in lbs/sec.; e_t, thermal efficiency; e_p, propulsive efficiency; and c, combustion energy per pound of fuel in ft-lbs. Our interest is in fuel consumption per hour, which is obtained by solving (2) for M_f, multiplying by 3600, and substituting for power its equivalent from equation (1). Thus

$$F = \frac{3600\left(\frac{b_1 s_1 + s_2}{2}\rho V^3 + \frac{2W^2}{b_2 s_1 \rho V}\right)}{c \cdot e_p \cdot e_t}, \tag{3}$$

where F represents fuel consumed in pounds per hour.

It would be highly desirable to spell out in quantitative terms the determinants of the independent variables in this equation. It is not possible to do so in the case of the numerator. The four elements, b_1, b_2, s_1, s_2, are aircraft characteristics determined in the construction of the airplane. They are influenced by economic considerations within the limits of the technical situation, but for the computation of airline costs they must be taken as data, given the type of aircraft. The determination of desirable values for these factors is one of the basic problems of aircraft engineering.

The speed and altitude of any particular operation are variable within wide limits at the discretion of the managing authority. Considerations of passenger comfort and convenience as well as of some costs enter the determination of the values used in practice. However, it is impossible to formalize the determinants of these variables since the technical limits are themselves by no means rigid and the considerations of passenger appeal are not readily quantified.

Much the same is true of weight. A range of operation is set by technical and institutional forces; within this range the actual gross weight varies in accordance with managerial policy and the amount of payload presented by the public.

Thus all the independent variables in (1) must be taken as data, determined either by the manufacturers of the aircraft or by the management of the airlines. Given these data the power required is determined, as in (1).

It is possible to indicate considerably more about the determinants of the terms in the denominator of (3). The energy content of the fuel, c, is largely determined by the state of technology in the petroleum industry and is, of course, a datum here.

The *thermal efficiency* is a factor which takes account of the failure of the fuel charge to deliver to the crankshaft the full mechanical equivalent of the heat energy contained in the fuel. It is possible to obtain a first approximation of thermal efficiency given only the fuel-air ratio and the compression ratio. The approximation is based upon highly unrealistic assumptions, but they can be removed subsequently and a reasonably satisfactory approximation of thermal efficiency obtained.

The compression ratio is the ratio of the maximum to the minimum interior volume of the cylinders. The greater this ratio, the greater the thermal efficiency; the ratio is limited primarily by the tendency of the fuel to detonate at high compression ratios [25, p. 148]. The compression ratio is a characteristic built into the engine.

The fuel-air ratio is the ratio of fuel to air (by weight) in the combustion charge in the cylinder. It is subject to pilot control. Usually the pilot can choose between a rich and a lean automatic setting or can adjust the mixture manually. The desirable fuel-air ratio is an engineering problem which is not entirely settled.

Given the fuel-air and the compression ratios the thermal efficiency of the

"fuel-air cycle" can be determined. The fuel-air cycle is a theoretical model based upon the following assumptions [25, pp. 31–32]:

> ... it is assumed that the medium (the charge) before combustion consists of a homogeneous mixture. ... It is further assumed that no heat is transferred across the boundaries of the medium during any part of the cycle and that chemical equilibrium exists at all times, ... combustion occurs instantaneously. ...

It is also assumed that the valves open and close instantaneously at top and bottom dead center, that the intake and exhaust pressures are the same and are constant, and that the engine is frictionless.

The fuel-air cycle does not encompass the use of throttles or superchargers, which can change the ratio of the intake to exhaust pressure. Although, theoretically, increased ratio of intake to exhaust pressure increases thermal efficiency, over the operating range of the ratio it appears to have little effect.[5]

The assumption that the gases in the charge are homogeneous is unrealistic in that the charge differs somewhat in composition between the various cylinders of a multicylinder engine and in that the charge in any cylinder is not perfectly mixed. There appears to be no general statement of the quantitative relationship between incomplete mixing and thermal efficiency.

The assumptions that combustion is instantaneous, that no heat is lost to the environment, and that there is no friction are obviously unrealistic. In an actual engine, well maintained, the thermal efficiency is reduced about fifteen per cent due to heat losses and the losses arising from the time combustion takes.[6] Friction losses in general increase with increased engine speed, although there are some other variables of secondary significance [25, pp. 187–205]. Thus engine friction is minimized by operating at the minimum engine speed at which the power required can be developed within the structural limitations of the engine. Mechanical friction reaches a maximum at take-off (maximum) RPM[7] at which speed it reduces thermal efficiency by a factor of about 0.9. It is convenient to summarize the determinants of thermal efficiency in a simple equation:

$$e_t = e_f \cdot e_c \cdot f(p_i/p_e) \cdot e_m, \tag{4}$$

where e_t represents thermal efficiency; e_f, thermal efficiency of the theoretical fuel-air cycle; e_c, a factor taking account of heat losses and those due to combustion time in an actual engine (about 0.85); $f(p_i/p_e)$, correction for the use of throttle and supercharger; and e_m, mechanical efficiency, an inverse function of friction (a minimum of about 0.90).

The *propulsive efficiency*, the last term in equation (2), is defined as

$$e_p = P/P_b, \tag{5}$$

where P_b represents brake power (delivered to the crankshaft). Clearly the entire power of the engine could be absorbed by the propeller without developing any thrust if the propeller blades were, in effect, parallel to their plane of rotation and simply sliced through the air or if they stood perpendicular to their plane of rotation with their broad sides merely pushing against the air. Between these two extremes the propeller reacts with the air to give thrust to the airplane. *Ceteris*

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[5] This conclusion is based upon some empirically derived charts provided the writer by Professor Edward S. Taylor.

[6] This figure was suggested as being representative by Taylor.

[7] Engine revolutions per minute.

paribus, there is one blade pitch (and one propeller speed associated with it) which maximizes the forward thrust for each level of engine power.

Propulsive efficiency, like thermal efficiency, is a rather complex variable. Four factors—altitude, velocity, brake power, and propeller speed (hence pitch)—affect it (given the shape of the propeller and the aerodynamic characteristics of the nacelle in which it is mounted) [**23**, pp. 207 ff., **15**, Chapters XI, XII].

Apparently it is impossible to formulate general quantitative statements of the determinants of propulsive efficiency. For the present purposes it is necessary to know only that in modern transport airplanes the propulsive efficiency can be held within about one per cent of the maximum value under virtually all conditions in normal flight (**23**, pp. 207 ff., **13**, p. 431]. Within the operationally significant range it is maximized by reducing engine speed to the minimum allowable value [**27**, p. 253]. With modern installations propulsive efficiency can be held within a narrow range of 0.85.[8]

Throughout the above discussion crude approximations have been used for the factors with which to correct the theoretical thermal efficiency of the fuel-air cycle and for propulsive efficiency. These can, of course, give only approximate results. It is naturally desirable to ascertain the precise values that should be used in any particular operation. I have been able to find no systematic presentation of the determinants of these values in general, and apparently the only way in which they could be ascertained in any particular case would be by empirically testing the equipment to be used. Nevertheless, the approximations used are of sufficient accuracy to give useful results.

The discussion of the determinants of fuel consumption for a single airplane in equilibrium flight is now completed. In order to apply equation (3) to any actual airline operation, several modifications must be introduced. First, since different stages in a flight employ different speeds and altitudes and since the weight of the airplane changes as the fuel is consumed, weighted averages of the pertinent variables must be employed. Furthermore, if different types of airplanes are used in a given operation, it is necessary to use weighted averages for the aircraft characteristics, s_1, s_2, b_1, and b_2; or, alternatively, the fuel consumed by each type of aircraft must be computed separately. Thirdly, an important amount of fuel is consumed in ground operations, taxiing, take-off and landing, and engine warm-up and inspection. The above equation is not applicable in the determination of the amount of fuel used in these operations because of the fact that the values assumed for the various efficiencies and corrections of them would not be realistic. The notions of lift and drag do not apply (in anything like their usual sense) to ground operations, where the speed may be negligible and friction of the landing gear may be of dominant significance. Therefore it is necessary to determine the fuel expended in this way empirically. Such empirical research is not within the scope of the present limited study.

Since most of the fuel used in ground operations (that used in taxiing, take-off and landing, and engine run-up) is directly associated with the number of landings in a particular operation, it is possible to add to (3) a term taking account of such fuel expenditure as a function of the number of landings. The rest of the fuel used in ground operations is taken as a quantity to be determined directly in each practical study. Thus

$$(6)\qquad F = \frac{3000H_a\left(\dfrac{b_1s_1 + s_2}{2}\rho V^3 + \dfrac{2W^2}{b_2s_1\rho V} - T\right)}{c \cdot e_p \cdot e_f \cdot e_c \cdot c_n \cdot f(p/p_a)} + B \cdot M_b + E,$$

[8] This value was mentioned in conversation by Taylor.

where B represents the number of landings; M_b, the fuel consumed per landing in associated ground operations; E, the fuel used in other ground operations; H_a, the number of hours flying time; and T, the power equivalent of the jet thrust obtained from the exhaust.

Equation (6) is a complete statement of the determinants of the real aircraft fuel costs of airline operations. The advantages of having a cost statement in this form lie in its generality and quantitative precision. The statement as presented cannot be affected by technological change; the form of the equation is entirely independent of the type of aircraft employed, the conditions of operation, or institutional factors. These influence only the values of the variables in the equation. Even in the case of so great a change as from the conventional reciprocating engines to jet engines, the form of the equation would be unchanged; the absence of a propeller would, of course, make $e_p = 1$. Operations with the equation designed to indicate the usefulness of this approach will be performed in Part III. Before that is done it is desirable to demonstrate that the equation is operational; this is the task of Part II.

It may be well to point out in passing that (3) has not been derived in a manner which relies upon airline operational experience (statistics) at all, but depends upon the principles of thermodynamics and aerodynamics. In equation (6) terms are added which take account of some of the "practical" aspects of airline operation. The values used in Part II (cf. Table I) were obtained, for the most part, from engineering and operating records which are kept for purposes other than fuel control.

II. APPLICATION OF THE FUEL EQUATION

In Part II the equation (6) just developed is tested to determine whether it is in fact operational. The validity of equation (3) in explaining the determinants of fuel consumption in flight is based upon physical principles which are not being tested here. Equation (6) contains additional terms, $B \cdot M_b$ and E, which can be defined to account for any fuel consumed other than in flight. Thus there is no problem of verifying the relationships established. However, some fuel evaporates, gets spilled, or is "lost" in other ways, and the data available on fuel consumption and its determinants may be inaccurate. The test is provided only to ascertain whether, with the available data, the above equation yields reasonably satisfactory results.

The test was conducted with the assistance of Northeast Airlines and was possible only because of the excellent cooperation received from many of the officials of that company.[9]

The technique used in testing the formula is merely to substitute in it values obtained from the airline for each of the terms in the equation. The fuel consumption predicted is then compared with the actual fuel consumption during the same period. Only fuel consumed in revenue flight is dealt with. Two separate computations were made; the fuel consumption for the month of June, 1949, was predicted

[9] I am deeply indebted to the president, Mr. George E. Gardner, for permitting me to obtain the necessary information and for his own valuable comments and suggestions. Mr. A. A. Lane, Vice President of Operations, took a very active interest in the project and discussed it at many stages and directed me to the people who could answer detailed questions. Mr. Edward Splaine and Mr. Howard Tufts, of the engineering department, and Chief Pilot Dixwell answered many of my questions and provided me with most of the data. Mr. H. Heard and Mr. D. W. H. MacKinnon both took an active interest in the study.

for each of the two major types of airplanes employed by the airline (the DC-3 and the CV-240). This was deemed superior to computing total fuel consumption in a single operation, which might well give a satisfactorily small error that would merely be the result of offsetting large errors. Northeast Airlines also employed a third type of airplane (the DC-4) but used it very sporadically during the month in question; therefore it was not considered that the data on actual fuel consumption were sufficiently reliable to include them in the computation.

The engineering department provided values for the airplane characteristics and the energy content of the fuel, b_1, b_2, s_1, s_2, and c; these were taken as constants for each type of aircraft. The first difficulty was encountered in determining an appropriate gross weight to employ. An estimate, simply a considered judgment, of typical gross take-off weights employed by each of the types of aircraft was obtained. These values were substantially less than the maximum allowable gross weights because of the fact that the payload carried is not always the maximum and the fields used do not always permit operation with maximum loads. In order to determine the amount that should be deducted from take-off weight to allow for the burning off of fuel, a brief inquiry was made into the fueling practices of the airline, and it was decided that a typical burn-off between fueling points for the DC-3 would be about 2400 pounds and for the CV-240 about 3000 pounds. Half of this amount was deducted from each typical take-off weight; the balance was taken as the typical en route gross weight. This figure was then considered a constant for each type of aircraft.[10] The crudity of this figure is apparent.

The number of hours flown by each type of aircraft was provided from their logs. This figure includes the time spent on the ground between the loading ramp and take-off and between landing and the unloading ramp. Since (3) is not suited to handling fuel consumed in ground operations, it is necessary to deduct the time so spent from total flying time as reported. Captain Dixwell, the chief pilot, provided the information that about 17% of pilot "flying" hours were spent on the ground; multiplying the total number of hours by this factor provided an estimate of total taxiing time for the month. An estimate of the average time spent per take-off and per landing was also obtained from the chief pilot.[11] The sum of these figures multiplied by the number of landings made during the month provided the total time spent in landing and taking off. The sum of the taxiing time and the landing and take-off time was deducted from total "flying" time as obtained from aircraft logs to get the total time to which the fuel consumption formula could be applied, H_k.

Because average altitude and speed depend upon whether the airplane is climbing, gliding, or cruising, it was necessary to get estimates of the amounts of time spent in each of these operations. Statements of the rate of climb or descent and the cruising altitude used by each type of aircraft were obtained. Dividing the difference between cruising altitude and the altitude at which normal climb was considered to begin and at which normal glide was considered to end by the rate of climb and the rate of descent, respectively, then multiplying each quotient by the number of stops gave the time spent in climbing and gliding, respectively. Deducting the sum of these from the total flying time previously computed gave the hours spent in level cruise.

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[10] Since the gross weight enters the fuel equation as a square, deducting one-half the burn-off is not an entirely sound procedure; but given the roughness of the computation it seemed satisfactory.

[11] The time spent in these operations was, for the sake of simplifying the arithmetic, defined as all time in the air not spent at normal cruise, normal climb, or normal glide power settings.

The computation of the value of ρ to be used is based upon the assumption that the density of the air varies as in the standard atmosphere.[12] The variation in the density with altitude under this assumption is readily obtainable. Taking the typical altitude suggested by the airline (5,000 ft.), a value of 0.002049 was obtained for cruising; taking 2,500 ft. as the typical altitude in climb and glide, a value of 0.002209 was obtained for these operations [11, Appendix]. The final value of ρ used in the computations was an arithmetic average of these weighted by the number of hours spent in cruising and in climbing and gliding, respectively. Estimates of typical speeds for both aircraft in each of the three flight attitudes were obtained from the chief pilot. These were converted to feet per second and an average, weighted as in the case of ρ, was obtained. Since the cube of the velocity appears in the equation, the speed used in each flight attitude was cubed and weighted in the same manner to find an average velocity cube to use for each aircraft. Thus all the values needed for the numerator in (3) were obtained.

The denominator was less difficult. The value of 0.85, suggested in Part I, was taken for e_p. Thermal efficiency of the fuel-air cycle, e_f, was computed for the appropriate compression ratios and the fuel-air ratios used in cruising and gliding, on the one hand, and in climbing, on the other. For mechanical efficiency, e_m, a value of 0.92 was taken for climbing (at relatively high engine speeds) and of 0.94 for gliding and cruising. An average, weighted by the hours in which each was employed, was used in the final computations. It was found that the use of superchargers and throttles did not significantly change the theoretical thermal efficiency.[13] The value of 0.85, suggested in Part I, was taken for e_c. Thus the values for equation (3) were obtained.

The power required in the CV-240 is reduced somewhat below that indicated by equation (3) by the fact that the exhaust stacks are so arranged as to provide some forward thrust. The thrust in pounds at the altitudes in question was provided by the engineering department of the airline: since the speeds of operation were known, it was possible to convert the thrust for climbing, gliding, and cruising into power equivalents. These were weighted in the usual fashion to get an average thrust, which was deducted from the power required of the propeller. The DC-3 does not have the exhaust thrust arrangement and hence it is not necessary to make this deduction.

The determination of the amounts of fuel consumed in landing and taking-off and in other ground operations presented many problems and was not entirely satisfactory. The airline furnished figures for the DC-3 for fuel consumed in landing, engine run-up prior to take-off, taking-off, and taxiing on a per stop basis, which, because of the great amount of experience with that airplane, were considered adequate empirical estimates. Multiplying by the number of landings gave $B \cdot M_b$. To this was added a figure for the daily warm-up and engine inspection, the only other ground operation taken into consideration. No figure for this was available from the airline. However, a value of 50 lbs per aircraft per day has been found by M. G. Beard [1, p. 56] to be appropriate. This figure, multiplied by

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[12] A model used in study of the atmosphere which assumes [15, p. 8]: "1. The air is a perfect gas with the gas constant . . . 53.33089 feet per degree F. 2. The pressure at sea level is . . . 29.921 in. Hg. 3. The temperature at sea level is . . . 59°F. 4. . . . the temperature gradient [is] . . . 0.003566°F. (per foot of altitude above sea level). . . ."

[13] Taking the prescribed manifold pressure for the intake pressure and the atmospheric pressure at each altitude plus 2 in. Hg as exhaust pressure. The two inches were added arbitrarily to take account of friction in the exhaust manifold.

the number of aircraft in use and the number of days in the month, is taken as the total value of the fuel consumed in the daily check and warm-up, E, for the DC-3.

For the CV-240, insufficient experience with the airplane made satisfactory empirical data unobtainable from the airline. Fuel consumed per minute in taxiing, estimated by R. Dixon Speas [22, p. 180] is 6 lbs.; this, times number of minutes spent in taxiing (determined as indicated above), was used for the total fuel consumed in taxiing. For fuel spent in actual landing, the same rate per minute was employed on the grounds that no direct estimate was available and that, in the actual landing operation, throttles are closed so that at this point fuel is probably used at considerably less than the taxi rate, while during the approach for landing some power is used, probably more than in average taxiing.

Data on the actual rate of fuel consumption for take-off power were available from the airline, 49 lbs/min., but no accurate estimate of time at take-off power was available. One and one-half minutes per take-off was taken as representative.

TABLE I

VALUES OF THE VARIABLES USED IN THE NUMERICAL TEST

	DC-3	*CV-240*
H_h	1135.2	724.5
$b_1s_1 + s_2$	22.05	19.6
b_2	28.70	32.34
s_1	987	817
ρ	0.00219	0.00213
V	228	303
V^3	12,800,000	35,700,000
W	22,500	36,500
C	14,587,500	14,587,500
e_p	0.85	0.85
e_c	0.85	0.85
$f(p_t/p_e)$	1.00	1.00
e_n	0.93	0.93
e_f	0.37	0.38
B	2,814	1,305
M_b	120	130
E	12,000	15,000
I	...	90,200

No data at all were available on the fuel consumed in warm-up and daily inspection, so twice the value of the DC-3 per inspection was taken simply because the maximum power of the CV-240 engine is twice that of the DC-3. The results of these computations are indicated in Table I. Note that a value only for the sum of the quantities $b_1s_1 + s_2$ is available. Substituting these figures in equation (6) yields predicted values for fuel consumption in lbs/hr.

The results of these computations and the comparison of the predicted with the actual total fuel consumption[14] for the month of June are indicated in Table II.[15] The total error is indicated in Table III as 5%. Given the fact that nearly all the data are estimates, some of them very crude estimates, the results are suspiciously good. The errors of 4% and 6% in predicting the fuel consumption of the two types

[14] The data from the airline were in gallons and were converted to pounds by multiplying by a factor of 6.

[15] The question of the number of significant figures in a complex series of computations as in the present problem is always difficult. The rounding-off used in Tables II and III is rather arbitrary but appears to be adequate for the present purpose.

of aircraft probably include some offsetting errors. However, the purpose of the present part has clearly been accomplished. The equation is operational and does produce a reasonably accurate approximation of fuel consumption within the limitations of the data. More accurate results must await more exhaustive empirical research.

Before going on to Part III it will be well to pause and indicate some of the chief sources of error in the computations and data. Undoubtedly the most dubious figures are the estimates of fuel consumed in landing and associated operations. It was not possible to ascertain how the airline had computed the figures for the DC-3, but the personnel in the engineering department felt that they were reasonably reliable because of the long experience with the airplane. Nevertheless, since roughly 40% of the fuel used is consumed in these operations, any substantial error in the figures per landing would cause a very marked change in the predicted result. The landing figures for the CV-240 are far less reliable than those of the

TABLE II

PREDICTED AND ACTUAL FUEL CONSUMPTION OF THE DC-3 AND CV-240

Fuel Consumed	*DC-3*	*CV-240*
In flight	427,000	634,000
In landing, taking-off, taxiing, and engine run-up	338,000	171,000
In warm-up and daily inspection	12,000	15,000
Predicted total	780,000	820,000
Actual total	809,322	870,966
Error	29,322	50,966
Per cent error	4	6

TABLE III

PREDICTED AND ACTUAL FUEL CONSUMPTION FOR BOTH AIRPLANES TOGETHER

Predicted total	1,600,000
Actual total	1,680,288
Error	80,288
Per cent error	5

DC-3 and are little more than reasonable guesses. Until considerably more empirical work is done on fuel consumption in ground operations, any formula of the type used here is subject to a wide range of error. There may be as large a relative error in the figure for daily warm-up and inspection, but it is of much less absolute significance.

The prediction of fuel consumed in flight is relatively reliable, but even here there is considerable room for error. The figures for altitude and speed might have been estimated more accurately by a laborious analysis of the flight logs of the aircraft. However, it is by no means certain that a reliable estimate of velocity could have been obtained even in that fashion. What was employed instead was merely the considered judgment of the chief pilot, based on his experience. Since fuel consumption varies as the cube of the velocity, errors in this term would cause considerable errors in the final prediction. The figure used for typical gross weight is also somewhat suspect. It is based upon two guesses, first, what the typical take-off weight is, and second, what the typical fuel burn-off en route is. The estimates of propulsive and thermal efficiency are also crude, as was indicated in Part I. The rest of the data is probably relatively respectable.

Lastly, it should be pointed out that an underestimation of fuel consumption is what one would expect. The actual fuel consumption figure was obtained from

service logs which merely record the amount of fuel poured into the airplanes. Some of this fuel is wasted in various ways, e g., spilling; some of it evaporates; some of it is used in maintenance checks other than the one included in the calculation. An empirical determination of the amount of fuel dissipated in these ways would be far more laborious than it is worth from the point of view of the economist. Since no attempt was made to estimate these quantities, the 5% error observed does not seem excessive.

III. THE MARGINAL COST FUNCTIONS

Before developing the marginal cost functions which are the chief aim of this paper, it is necessary to digress briefly upon the nature of output. For the present, output of an airline will be measured in gross ft-lbs produced per month. Although for most purposes this is not a satisfactory measure, it is simplest in view of the units in equation (6) and can be converted readily, as necessary, to more conventional measures. The quantity of output is expressed as

$$X = 3600H_h \cdot V \cdot W. \tag{7}$$

One of the difficulties encountered in the conventional derivation of cost functions by statistical techniques [cf. **6, 29**] is that, historically, changes in quantity of output are frequently associated with qualitative changes. The techniques commonly used to isolate quantitative changes usually, perhaps inevitably, impair the final results. This problem appears to stem from an addiction to treating output as a unidimensional quantity (in accordance with classical theory) and to trying to hold constant the other factors which are known to influence the relationship between product and costs. It is theoretically possible to treat product as being multidimensional with one dimension for each significant[16] product characteristic, relating changes in each dimension to cost. This is the approach of the present paper.[17]

In the case of many service industries (and perhaps some others) peculiar problems arise since output is measured as the (arithmetic) product of two quantities which may themselves have significant qualitative aspects. Examples of such measures of product are kilowatt-hours, ton-miles, passenger-miles. Clearly one kilowatt produced for one thousand hours is qualitatively entirely different to the buyer (as well as to the supplier) from a thousand kilowatts produced for one hour. The analogue holds in the case of ton- and passenger-miles. In such a situation the isolation of qualitative and quantitative changes requires

[16] In the case of a cost analysis, "significant" is defined to include only those product characteristics which influence cost.

[17] The problems of product differentiation have been treated widely since Professor Chamberlin's basic work, but of particular interest in the present context is Hans Brems' paper [2].

ad hoc determination of the product characteristics which are qualitatively significant to the buyers. Changes in the quantity of output, as defined, which do not involve these qualitative elements if there are such, can be taken as isolated quantitative changes.

The remainder of the paper is devoted to the derivation of marginal cost functions associated with changes in the quantity of output, changes in the quality of product, and changes in the technology of production. What is called here marginal cost is only the marginal cost of fuel.[18] At least some of the changes covered would involve some inputs other than fuel, but the equations below show only the increment of fuel required. Thus they are not "total marginal cost" functions but only a part of the marginal cost. The computation of "total marginal cost" would require a similar analysis of all the inputs employed.

Further, it is clear that the marginal cost derived is a real, not a money cost, function, i.e., it is expressed in pounds of fuel. Multiplying by price per pound would provide a money marginal cost of fuel.[19]

However, it should be stressed that these cost functions *do not* depend upon any assumptions of constancy (relative or absolute) of the combination of the factors of production. Since the equation developed above includes *all* factors that can influence fuel consumption in flying operations, it is capable of taking account of any change in fuel consumption, whatever its ultimate source.[20] However, it is true, of course, that in determining the impact of each separate change the assumption of *ceteris paribus* is retained; effects of simultaneous changes in more than one variable can be handled by combining the pertinent equations.

Marginal Cost of Quantitative Changes

From equation (7) it is clear that changes in output in air transportation can be affected by increasing speed, weight, or hours.[21] A change in speed constitutes changing the basic product characteristic of air transportation. Changes in weight and hours, *ceteris paribus*, can be taken

[18] This marginal cost concept is not the same as the "coefficient of costicity" developed by Bréguet [19] since the latter deals with total unit cost and is really the elasticity of the average cost function with respect to its various determinants. "Per cent variable," a cost concept developed by Ford K. Edwards, is similar to the "coefficient of costicity." For Edwards' concept see [26, p. 27].

[19] In the case of a monopsonistic price the problem is empirically more complex.

[20] The above does not imply that the fuel input coefficient is independent of other inputs but that changes in other inputs can affect that coefficient only through the impact upon the independent variables in equation (6). The necessity for making this clear was pointed out by Mr. George H. Borts of The University of Chicago.

[21] Changes in altitude, ρ, could affect the quantity only through one of these terms.

as being changes in only the quantity of product. A change in hours with no change in the quality of product would require that both speed and the number of landings per hour remain fixed.[22] Therefore, in the present context, we must assume that $B = kH_h$. Then the partial derivative of fuel consumption with respect to hours is

$$(8)\qquad \frac{\delta F}{\delta H_h} = \frac{3000\left(\frac{b_1 s_1 + s_2}{2}\rho V^3 + \frac{2W^2}{b_2 s_1 \rho V} - T\right)}{c \cdot e_p \cdot e_t} + k \cdot M_b.$$

Although the fuel consumption in landing[23] will almost certainly change somewhat with changes in gross weight of the airplane, there is no way of determining formally the relationship between these factors. Ignoring the effect upon fuel consumed in landing, therefore, we get

$$(9)\qquad \frac{\delta F}{\delta W} = \frac{(4)(3000)H_h}{b_2 s_1 \rho V} \cdot \frac{W}{c \cdot e_p \cdot e_t}.$$

If output of a given quality is measured in gross pound-miles per month, then the two above equations indicate the marginal cost of increasing the quantity of output in either of the two ways possible. But life is not quite so simple. The management does not always have the choice of increasing output by whichever of these means will be more economical. The time distribution of demand partly determines the manner in which output must be increased. An increase in demand which falls at a time when aircraft are flying with less than full payloads will permit an increase in W (average gross weight per airplane). An increase in demand which falls either at times when no airplanes are flying the desired run or when they are flying fully loaded is a demand for more flying hours. So even here it is a little difficult to draw a firm, fine line between quantitative and qualitative changes in output.

The marginal cost, i.e., the marginal real cost of fuel, is linear in the case of changes in hours, but in the second case it is curvilinear, cost increasing as the square of the weight. Here is an empirically derived marginal cost curve with the upward concavity of the elementary texts! The fact that it has been derived here for gross weight, whereas weight of payload is usually more significant in measuring output, is of no significance since, *ceteris paribus*, gross weight increases by the same amounts as payload.

[22] Changes in hours with the absolute number of landings fixed would involve changes in speed or routes.

[23] Hereafter the word "landing" will encompass all operations included in the computation of $B \cdot M_b$.

Marginal Cost of Qualitative Changes

The most important product characteristic of air transportation is speed. Substituting from equation (7), $x/(3600W \cdot V)$ for H_h,

$$(10) \qquad \frac{\delta F}{\delta V} = \frac{x}{c \cdot e_p \cdot e_t \cdot W}\left[(b_1 s_1 + s_2)\rho V - \frac{4W^2}{b_2 s_1 \rho V^3} + \frac{T}{V^2}\right].$$

Here again is a curvilinear marginal cost function.

Changes in the number of landings may constitute a change in the quality of product, the marginal cost of which is a simple linear expression,

$$(11) \qquad \delta F/\delta B = M_b .$$

Altitude is an important product characteristic since it influences smoothness of the flight and, in warm weather, cabin temperature. In practice, changes in altitude may be associated with changes in V because the range of speeds possible, within the limits set by the stall characteristics of the airplane and the power characteristics of the engine, varies with altitude. Also, the time spent in climbing and gliding (which influence the average speed, which appears in the formula) would change with altitude if the rate of climb remained constant. However, it is possible within the power limitations of the engine to keep these times and cruising speed constant as altitude changes. Therefore, within limits, which may be rather narrow in practice but which are impossible to define without considerable investigation into aerodynamics and thermo dynamics,[24] speed can be held constant while altitude varies.

On this assumption

$$(12) \qquad \frac{\delta F}{\delta \rho} = \frac{3C00H_h}{c \cdot e_p \cdot e_t}\left(\frac{b_1 s_1 + s_2}{2} V^3 - \frac{2W^2}{b_2 s_1 \rho^2 V}\right).$$

Thus it has proven possible to quantify one cost of changing product characteristics, and the marginal cost measured in the appropriate dimensions is linear in one of the cases studied and curvilinear in two.

Marginal Cost as a Function of Technical Change

All other elements in (6) can be taken as technical conditions of production which can be changed without changing either the quality or quantity of product. The derivatives of these factors are readily obtained and are listed for completeness.

Changes in airplane structure or design:

$$(13) \qquad \delta F/\delta s_1 = \lambda[(b_1 \rho V^3/2) - (2W^2/b_2 \rho V s_1^2)],$$

[24] In fact there is, as far as I know, no realistic formal solution of this problem.

(14) $$\delta F/\delta s_2 = \lambda(\rho V^3/2),$$

(15) $$\delta F/\delta b_1 = \lambda(s_1\rho V^3/2),$$

(16) $$\delta F/\delta b_2 = -\lambda(2W^2/s_1\rho V b_2^2),$$

(17) $$\delta F/\delta T = -\lambda,$$

where $\lambda = 3600\, H_h/c\cdot e_p\cdot e_t$.

Changes in engine performance and associated variables:

(18) $$\delta F/\delta c = -3600\, H_h P_r/c^2\cdot e_t\cdot e_p,$$

(19) $$\delta F/\delta e_t = -3600\, H_h P_r/c\cdot e_t^2\cdot e_p,$$

(20) $$\delta F/\delta e_p = -3600\, H_h P_r/c\cdot e_t\cdot e_p^2,$$

(21) $$\delta F/\delta M_b = B,$$

(22) $$\delta F/\delta E = 1,$$

where $P_r = [(b_1 s_1 + s_2)/2\rho V^3] + [2W^2/b_2 s_1 V\rho] - T$.

These equations give the marginal cost of fuel of operating under different technical conditions, i.e., they show the fuel saving or expense of changing the technical conditions of production. Thus they show the value of technical or operational innovations, and hence (a) the ones that may (given the marginal cost of introducing the innovations) be economically introduced under any given technological situation, and (b) the directions in which research might be most profitably directed to change the technological situation.

Changes in these factors do not necessarily require technological change in the sense of invention or advances in pure science, although all of them could be brought about as a result of incorporating such changes. Some may be associated with the improvement of managerial practices, for example, changes in e_t, e_p, M_b, or E.[25] The introduction of a new type of aircraft will involve changes in many of the factors simultaneously, but in the designing and development of aircraft each can be varied independently of the others. Even given the type of aircraft, many of them can be varied independently of the others. Of course, the introduction of a new aircraft may also change the values of V and W and perhaps the permissible values of ρ.

Conclusion

The advantages of this approach to cost problems is fairly clear. First, it does not abstract changes in the techniques of production,

[25] In the technical jargon, a management may be operating off an isoquant in that, given the technological situation, more fuel than is required for the amounts of the other factors employed is being expended. A managerial change to economise fuel would constitute an approach to the isoquant.

factor prices, product quality, and product mix as do the usual statistical techniques. Instead, it completely explains the consequences of changes in these variables. For example, the marginal cost in fuel of every type of product modification can be computed from (6), since any qualitative change that does not affect the independent variables in the equation has a marginal cost of fuel equal to zero. The analogue holds for technical change. Second, it does not depend upon fitting to past operational and cost data curves whose form has little rationale. The method depends upon being able to obtain objective statements like equation (6). Nevertheless, it is clear that the present approach must be supplemented by—or is supplementary to—the statistical investigations in that the phenomena observed in the conventional approach must be explainable with the present technique and in that there may be some areas of productive activity where adequate statements of physical interrelationships are not obtainable.

The limitations are equally clear. The amount of effort that must be expended to obtain a quantitative statement of the determinants of each type of input in a complex industry is very great. For some types of inputs it may be impossible to do so at all.

A complete statement of the determinants of airline cost functions would require a similar analysis of all the inputs used. With these analyses it would then be possible to determine the total real cost of any change in output, technology (combination of the factors), product quality, or product mix. Applying the money prices of the factors of production would complete the information required for the determination of the money cost of any change in the production of airline services. Of course, before the desired static combination of inputs could be ascertained, it would be necessary to go still further and obtain estimates of the marginal revenue of each product (quality or quantity) variation. The actual optimum combination of the factors depends, furthermore, upon expectations. Although there is no explicit treatment of expectations in the present method, it does provide a means of evaluating the cost of adjusting to any sort of expectational pattern.

Thus the chief disadvantage of the method is the large amount of effort required to carry it out. However, it appears to be a much more reliable approach to cost problems, and effort directed along these lines will at least be moving toward accurate and relatively unambiguous results. The typical statistical studies based upon unrealistic abstractions from change and burdened with their usual linear bias [cf. **6, 20, 24**] are easier means of obtaining *an* answer, but it appears necessary to seek rigorous rather than readily available conclusions.[26]

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STOCK AND FLOW ANALYSIS IN ECONOMICS

By Lawrence R. Klein

It is not always irrelevant whether one deals with stock or flow variables in economic analysis. Some simple dynamic models are shown in which the choice between stock and flow variables becomes essential. It is also not true that the liquidity preference theory of interest is identical with the loanable funds theory of interest. This problem is intimately connected with the distinctions between stock and flow analysis. These two interest theories are shown to be different in simple dynamic models.

W. Fellner and H. M. Somers have overlooked[1] some of the important differences between the analysis of stocks and of flows in monetary interest theory. It is correct but unenlightening to claim that, if the proper mathematical transformations are used, the relationships describing the economic system may be written either in terms of flows or of stocks. There are, however, some tricks involved, and it would be misleading simply to write off flow analysis and stock analysis as equivalent.

To fix ideas on a more precise footing, I shall proceed immediately to a simple mathematical model that contains the bare essentials of the matter at hand. I intend to show that there exist models in which the arguments of Fellner and Somers are incorrect, but I do not claim that there are no models in which they are correct, although I do consider the latter models trivial in some essential features.

Let us consider an aggregative economy in which the subjects can hold goods, perpetuities, or cash. Perpetuities are issued by the business firms and held by the households. The government controls the total issue of cash, which can be held by households or firms. Interest payments are calculated on the volume of securities held at the beginning of an accounting period. Accumulation of real capital will be ignored. The notation is as follows: n_t = employment (flow during period t), x_t = goods (flow during period t), B_t = market value of perpetuities (stock at the end of period t), M_{1t} = household cash (stock at end of period t), M_{2t} = business cash (stock at end of period t), w_t = wage rate (average during period t), p_t = price of goods (average during period t), r_t = interest rate (average during period t).

[1] W. Fellner and H. M. Somers, "Note on 'Stocks' and 'Flows' in Monetary Interest Theory," *Review of Economics and Statistics*, Vol. 31, May, 1949, pp. 145–146.

Editor's Note—In addition to the above reference, the reader's attention is invited to the following: W. Fellner and H. M. Somers, "Alternative Monetary Approaches to Interest Theory," *Review of Economics and Statistics*, Vol. 23, February, 1941, pp. 43–48; and Lawrence R. Klein, *The Keynesian Revolution*, New York: The Macmillan Co., pp. 117–123. See also the four contributions which follow the present paper.

The equations of the system are

$$(1)\qquad x_t^D = f_1(w_t/p_t, r_t)\text{, household demand for goods,}$$

$$(2)\qquad x_t^S = f_2(w_t/p_t, r_t)\text{, business supply of goods,}$$

$$(3)\qquad n_t^D = g_1(w_t/p_t, r_t)\text{, business demand for labor,}$$

$$(4)\qquad n_t^S = g_2(w_t/p_t, r_t)\text{, household supply of labor,}$$

$$(5)\qquad M_{1t}^D/p_t = h_1(w_t n_t^S/p_t, r_t)\text{, household demand for cash,}$$

$$(6)\qquad M_{2t}^D/p_t = h_2(p_t x_t^D/p_t, r_t)\text{, business demand for cash,}$$

$$(7)\qquad M_{1,t-1} + (1 + r_t)B_{t-1} + w_t n_t^S - p_t x_t^D = M_{1t}^D + B_t^D\text{,}$$
household budget restriction,

$$(8)\qquad M_{2,t-1} - (1 + r_t)B_{t-1} - w_t n_t^D + p_t x_t^S = M_{2t}^D - B_t^S\text{,}$$
business budget restriction,

$$(9)\qquad M_{1t}^D + M_{2t}^D = k = \text{constant, condition of stock equilibrium,}$$

$$(10)\qquad x_t^D = x_t^S\text{, condition of flow equilibrium,}$$

$$(11)\qquad n_t^D = n_t^S\text{, condition of flow equilibrium.}$$

In this system of eleven equations there are eleven variables as of time t: x_t^D, x_t^S, w_t, p_t, r_t, n_t^D, n_t^S, M_{1t}^D, M_{2t}^D, B_t^D, B_t^S. I shall assume that a unique, classical solution exists for this system, given the lagged values of M_1, M_2, and B. This procedure disregards, obviously, the possibilities of inconsistency which I look upon as the basis for the modern theory of employment and have discussed elsewhere. This accords also with my view of the de-emphasis of liquidity preference in the theory of employment.

This model is presented as a framework for analysis at the static level without further justification. It represents my version of classical economics except possibly for the fact that r_t appears as a variable in (5) and (6), but that is necessary in order to deal with the problem at hand. The question may legitimately be raised whether or not this classical model could be derived from the classical theories of utility and profit maximization in the household and business sectors of the economy. I shall only indicate an answer to this question without going into the formal details of derivation. Let us regard an individual consumer as maximizing a utility function depending upon a stream of goods, services, cash, and securities over a planning period and subject to a budget constraint. The budget constraint suitable for the maximization process will not be (7); it will instead be

$$\sum_{t=1}^{T} \frac{w_t n_t^S - p_t x_t^D}{(1 + r_1) \cdots (1 + r_t)} = 0,$$

which is obtained by discounting the sum of (7) over the period $1, 2, \cdots, T$ and assuming that the capital value of the household plan is zero; i.e., that the discounted stream of first differences in each period's cash holdings and the difference between the initial stock of securities and the discounted final value are zero. Plans are laid anew for each accounting period so that the demand-supply equations for variables carrying a subscript $t = 1$ become the effective demand-supply equations (1) and (4), and the effective budget restriction is (7). If the capital value of the plan is not zero, the equation system must be modified by allowing initial (lagged) stocks of wealth to appear as variables in (1)–(6). A similar scheme can be developed for the equations in the business sector of the economy.

From what I have presented thus far, it is not possible to say that the interest theory imbedded in (1)–(11) is based on the equation of supply and demand for cash or of supply and demand for securities. The demand equations for cash are (5) and (6); they are equated to the supply of cash in (9). The demand equation for securities is obtained by substituting (1), (4), and (5) into (7). The supply equation of securities is obtained by substituting (2), (3), and (6) into (8). If (9), (10), and (11) hold, then the demand for securities must equal the supply of securities. Hence a unique set of equilibrium market variables (w_t, p_t, r_t) is consistent with the two propositions that the supply and demand for the stock of cash are equated and that the supply and demand for the stock of securities are equated. I need not emphasize the emptiness of this proposition.[2]

The demand-supply equations for securities implied by (1)–(11) involve stock concepts. Since this model is a formalization of the Fellner-Somers model, I cannot understand why they insist that their supply-demand relation for securities (claims) is clearly a flow concept. Only if the derived demand-supply equations could be written with $B_t - B_{t-1}$ as a function of market variables alone is it reasonable to call them flow concepts.

I (and others) have really meant something more than is contained in the above model when I say that the determination of the interest rate is derived from the theory of liquidity preference. The static equilibrium model of (1)–(11) is only an abstraction (of high order) derived

[2] Since the interest and real wage rates can be obtained from (1)–(4) and (10)–(11), Hicks's time-honored demonstration of the equivalence of the liquidity preference and loanable funds theories makes little sense. The interest rate is determined in the real sector of the economy, where cash and securities can *both* be ignored, provided initial stocks of wealth are not included in (1)–(4) as indicated above. This classical approach does not rule out the concept of liquidity preference; it merely shows that this concept fits in with conventional equilibrium theory as well as with the modern theory of employment.

from a closer counterpart of the true economic process. I shall only invoke the classical "law of supply and demand" to demonstrate what I mean by a liquidity-preference theory of interest. My version of a hypothesis, which can be tested against the facts, is the following:

> When the supply of cash exceeds the demand for cash the interest rate falls, and when the demand for cash exceeds the supply of cash the interest rate rises.

An additional specification of this nature is absolutely necessary in order to give some content to the idea that the interest rate is a variable which brings about an equation of the supply and demand for cash. In place of equilibrium condition (9), I shall write the dynamic equation

$$r_t - r_{t-1} = h_3(k_t - M_{1t}^D - M_{2t}^D), \tag{9a}$$

with the property $0 = h_3(0)$; therefore (9) is consistent with (9a). In the dynamic system k_t is an exogenous variable. It is not necessarily a constant. It should also be stressed that k_t includes the excess reserves of the banking system. With a precious-metal reserve of G_t and a reserve ratio of λ, one may look upon $(1/\lambda)\, G_t$ as the potential supply of cash.

The routine versions of the dynamic replacements for (10) and (11) are

$$p_t - p_{t-1} = f_3(x_t^S - x_t^D), \qquad 0 = f_3(0), \tag{10a}$$

$$w_t - w_{t-1} = g_3(n_t^S - n_t^D), \qquad 0 = g_3(0). \tag{11a}$$

In the dynamic model (1)–(8), (9a)–(11a), I have a liquidity-preference theory of interest, and I do not have a supply-demand-for-securities theory of interest. By this remark I mean that there is not, in the model, a unique relation between the excess demand for securities and fluctuations in the rate of interest. The algebra of the situation leads to an excess demand relation for securities of the form

$$\begin{aligned} B_t^D - B_t^S = h_3^{-1}(r_t - r_{t-1}) + (M_{1,t-1} + M_{2,t-1} - k_t) \\ + w_t g_3^{-1}(w_t - w_{t-1}) + p_t f_3^{-1}(p_t - p_{t-1}). \end{aligned} \tag{12}$$

Unless the sum of terms of the right-hand member after the first vanish identically in t, the relation between excess demand for securities and interest fluctuations does not hold.

Should I suppress (9a) and use instead

$$r_t - r_{t-1} = h_4(B_t^S - B_t^D), \tag{9b}$$

I should no longer claim that my model contains a liquidity-preference theory of interest.

I have followed the conventional treatment of the "law of supply and demand," but the more general possibility

$$\text{(9c)}\quad h(k_t - M_{1t}^D - M_{2t}^D,\ p_t - p_{t-1},\ w_t - w_{t-1},\ r_t - r_{t-1}) = 0,$$

$$\text{(10c)}\quad f(x_t^S - x_t^D,\ p_t - p_{t-1},\ w_t - w_{t-1},\ r_t - r_{t-1}) = 0,$$

$$\text{(11c)}\quad g(n_t^S - n_t^D,\ p_t - p_{t-1},\ w_t - w_{t-1},\ r_t - r_{t-1}) = 0,$$

should not be overlooked. Instead of assuming that a single market price moves to wipe out a maladjustment between demand and supply, it seems preferable to assume that several market prices move to eradicate the maladjustment. Since many items are interrelated in demand or supply, it is not absolutely necessary that an item's own price must be called upon to perform the entire adjustment process. In a model consisting of (1)–(8) and (9c)–(11c), there is not enough information to characterize it as containing a liquidity-preference theory of interest rather than a supply-demand-for-securities theory.

The difference between flow analysis and stock analysis can be brought out clearly in this connection. Let the demand for any economic quantity during a period of time be denoted by y_t^D and the supply by y_t^S. The time dimension of these variables is such that they are flows. The average price at which they are traded during period t will be written as q_t. It is one theory to use the equation

$$q_t - q_{t-1} = F(y_t^S - y_t^D), \qquad F(0) = 0,$$

and quite another to use

$$q_t - q_{t-1} = F^*\left[\sum_{\tau=-\infty}^{t} (y_\tau^S - y_\tau^D)\right], \qquad F^*(0) = 0.$$

One relation assumes an adjustment to flow equilibrium and the other to stock equilibrium. In the continuous case it is well-known that these two theories lead to entirely different results.[3] It makes a good deal of difference in the Fellner-Somers theory whether the rate of interest is a fluctuating variable that tends to equilibrate the supply and demand for the outstanding stocks of securities or for the flow of securities during a period of time. For example, they must state unambiguously whether they view

$$\text{(13)}\quad r_t - r_{t-1} = G(B_t^S - B_{t-1}^S - B_t^D + B_{t-1}^D), \qquad G(0) = 0,$$

or

$$\text{(14)}\quad r_t - r_{t-1} = G^*(B_t^S - B_t^D), \qquad G^*(0) = 0,$$

[3] See, e.g., P. A. Samuelson, "The Stability of Equilibrium: Comparative Statics and Dynamics," Econometrica, Vol. 9, April, 1941, pp. 107–108, 112–113.

as valid relationships, in case they support the supply-demand-for-securities theory of interest. This point they have failed to realize.

It is something of a generalization to go from static equilibrium to a dynamic model, but it is still unsatisfactory. In econometric work one is never permitted to disregard the fact that the model has a stochastic structure. The particular sort of stochastic structure that is almost always assumed is one that requires clear thinking on the matter of stocks vs. flows. The choice of a method of statistical estimation depends on whether one assumes that people behave so as to demand a *stock* of cash in relation to certain market variables and income plus a nonautocorrelated random error or whether behavior is such that people demand a *flow* of cash in relation to certain market variables and income (or differences of these variables) plus a nonautocorrelated random error.[4] One must first decide whether fundamental economic decisions are made in terms of stock or flow variables in each particular case.

A final remark concerns a statement of "the obvious" by Fellner and Somers. They say:

> But it surely is evident without further argument that the value of a variable cannot equate the demand for anything with the supply of that thing *over a period of time*, unless it also accomplishes the equality of the willingness to hold that thing and the existing stock of that thing at any point of time during that period!

Let the continuous variable $z(t)$ denote the flow of excess demand for z at instant t. Surely they would not claim that a price system securing $\int_T^{T+1} z(t)\,dt = 0$ automatically secures $\int_{-\infty}^{T+\theta} z(t)\,dt = 0$ for all θ, $0 \leqslant \theta \leqslant 1$. Even the stronger assumption of continuous market clearance of the flow over the whole interval $(T, T+1)$, i.e., $z(t) = 0$, $T \leqslant t \leqslant T+1$, does not imply the clearance of the market for stocks unless $\int_{-\infty}^{T} z(t)\,dt = 0$.

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[4] The choice between a stock or a flow formulation affects the autocorrelation properties of the random errors in an obvious way. It is, of course, possible that neither formulation leads to nonautocorrelation.

STOCK AND FLOW ANALYSIS: COMMENT

BY WILLIAM FELLNER AND HAROLD M. SOMERS

DR. KLEIN sets up a model[1] which includes among its variables the rate of interest. In the static case, he finds that the results as to the relation between the loanable funds and liquidity preference theories are empty. In the dynamic case, he finds that the loanable funds theory does not appear at all. He concludes that we were in error in claiming that the two theories were equivalent in general and he traces our error to a confusion of stocks and flows.

Klein's statements concerning the difference between stock and flow analysis stem from his particular definition of stock analysis. On his definition of dynamic stock analysis, it is true that stock analysis is different from flow analysis. Definitions which are more usual in interest theory lead to the conclusion that stock and flow analyses are equivalent. Legitimate differences of opinion may, however, arise as to the relative usefulness of alternative definitions. After a brief discussion of this point, we will show that Klein is in error when he maintains that the "dynamic stock analysis" version of the loanable funds theory is essentially different from the corresponding version of the liquidity preference theory even if *his* concept of dynamic stock analysis is used. The relationship between corresponding versions of the two theories is, of course, the real issue in the present controversy. To anticipate our main point here, we may say that Klein disregards the fact that an excess demand for money can affect the rate of interest only via demand and supply in the securities markets.

In considering stocks at the end of any period t, we assumed: $\sum_{-\infty}^{t-1}(y_i^D - y_i^S) = 0$, where y^D is the demand for anything and y^S is its supply. In other words we assumed that equilibrium was established at the beginning of period t, i.e., the end of period $t - 1$, and that there is a given amount, say y_{t-1}, of the good at the beginning of the period. In Klein's definition of "stock analysis," however, $\sum_{-\infty}^{t-1} (y_i^D - y_i^S) \neq 0$ or, perhaps, $\sum_{-\infty}^{t-1}(y_i^D - y_i^S) \gtreqless 0$. In the latter formulation, our assumption would be considered just a special case.

The relevant question, then, is whether our definition of "stock analysis" is an ordinary or even a tenable one, especially in the context of interest theory. It will probably be generally agreed that in Lord Keynes's presentation there exists no excess demand at any point of time. If we refer to Lerner's article in the *Review of Economic Statistics* in

[1] Lawrence R. Klein, "Stock and Flow Analysis in Economics," ECONOMETRICA Vol. 18, July, 1950, pp. 236–241.

May, 1944,[2] we find that he uses exactly the same concept as we do. He adds the stock of money (not the sum of all preceding excess demands) at the beginning of the period to the monetary flows of the period to get the liquidity preference theory. We believe that this is the usual concept of "stock analysis" and the usual understanding of the liquidity preference theory. Lerner had previously used this approach in his article in the *Economic Journal* in 1938.[3] Klein criticizes Lerner's article in his *Keynesian Revolution*,[4] but not on this particular point.

Even though the issue is merely one of definition, there is nevertheless a question whether the scarce term "stock analysis" should be allocated to Klein's approach or to ours. Perhaps his is an "infinitely dynamic stock analysis" since it determines the price of the current period by taking explicit account of an infinite number of lagged excess demands.

Using our definition of "stock analysis," the relation between the demand and supply of the stock of securities in period t is equivalent to the relation between the respective flows during the period. Klein asks us to state unambiguously whether we view his equation (13) or (14) as valid relationships. The first of these may be rewritten

$$r_t - r_{t-1} = G\{(B_t^S - B_t^D) - (B_{t-1}^S - B_{t-1}^D)\}.$$

On the assumption previously stated, namely, that equilibrium was established at the end of period $t - 1$, we have $(B_{t-1}^S - B_{t-1}^D) = 0$. Hence equation (13) reduces to the same form as (14).

The above, however, disregards the main weakness of Klein's dynamic analysis. Through a process of substitution he derives an excess demand relation for securities [his equation (12)] , and then says, "Unless the sum of terms of the right-hand member after the first vanish identically in t, the relation between excess demand for securities and interest fluctuations does not hold." In this equation k_t is an exogenous variable and must be assumed equal to k_{t-1} in the absence of outside information to the contrary.

For convenience, let us rewrite the derived equation (12) as follows:

$$\begin{aligned} r_t - r_{t-1} = h_3[(B_t^D - B_t^S) - (M_{1,t-1} + M_{2,t-1} - k_t) \\ - w_t g_3^{-1}(w_t - w_{t-1}) - p_t f_3^{-1}(p_t - p_{t-1})]. \end{aligned} \tag{12.1}$$

According to this derived equation, changes in the rate of interest are functionally related, not only to the excess demand for securities, but

[2] "Interest Theory—Supply and Demand for Loans or Supply and Demand for Cash," Vol. 26, pp. 88–91.

[3] "Alternative Formulations of the Theory of Interest," Vol. 48, pp. 211–230.

[4] New York: The Macmillan Co., 1947, pp. 119–121.

also to a number of other economic variables. Using Klein's equations (10a) and (11a), the derived equation becomes

$$r_t - r_{t-1} = h_3[(B_t^D - B_t^S) - (M_{1,t-1} + M_{2,t-1} - k_t) - w_t(n_t^S - n_t^D) - p_t(x_t^S - x_t^D)]. \tag{12.2}$$

Thus we see that in Klein's derived equation changes in the current rate of interest are functionally related to the excess demands for (a) securities (stock at end of current period, which may be translated into a flow during the current period, as indicated above); (b) cash (stock at end of previous period, since k_t is assumed equal to k_{t-1}); (c) employment (flow during current period); and (d) goods (flow during current period).

All demands and supplies are expressed in monetary terms. Hence we can say that the change in the rate of interest is related to the aggregate monetary demand and supply. In our previous analysis[5] we showed diagrammatically that on a two-dimensional basis, i.e., where the rate of interest is the only variable to be determined and the prices of goods and services are given, the excess demands for (c) and (d) must be assumed to have been eliminated through adjustments in the appropriate prices (of goods and personal services). The excess demand for (b) must have been eliminated at the end of the previous period. Thus, in the two-dimensional case, the right-hand term after the first must actually be assumed to be zero.

The main question is how the two theories are related in a multidimensional dynamic system such as the one Klein sets up. If we look at the derived equation (12.1), we see that there are alleged to be some factors other than the demand and supply of securities that affect the rate of interest. It is true, of course, that in a multidimensional system there are a great many factors that affect the interest rate. However, in *any* system, these factors can affect the market rate of interest only through their effect on the demand and supply of interest-bearing securities. We are reminded of Lerner's assurances on this point: (1) "It cannot be denied that the rate of interest, being the price paid for a loan, must be at the level where the demand for loans is equal to the supply of loans," and (2) "In saying that the 'cash' theory of interest is preferable to the 'loans' theory, I do not deny that the actual rate of interest is in fact agreed upon by the suppliers and demanders for loans."[6] Yet the form of Klein's derived equation leaves us no alternative but to conclude that there is some method whereby economic factors can achieve changes in the rate of interest without influencing the demand

[5] "Alternative Monetary Approaches to Interest Theory," *Review of Economics and Statistics*, Vol. 23, February, 1941, pp. 43–48.

[6] Lerner, "Interest Theory," *op. cit.*, pp. 88, 90.

and supply of securities! Klein's equation (12.1) gives the securities market some role in determining the interest rate and yet provides a means of circumventing the securities market and affecting the interest rate directly. Even with equilibrium in the securities market there could be a change in the rate of interest. Klein's rate of interest thus becomes what has been called in a related context, "a grin without a cat."

The conclusion is that we are not free to write Klein's equation (9a) unless a unique relationship exists between the excess demand for money and the excess demand for claims, that is, between $k_t - M^D_{1,t} - M^D_{2,t}$ and $B^S_t - B^D_t$, with the result that the sum of all right-hand terms of (12.2) may alternatively be written as some function of $B^S_t - B^D_t$. Given such a functional relationship there exists no essential difference between using equation (9a) or using (9b). An analogous conclusion holds regardless of the degree of dynamics (in Klein's sense) which we wish to introduce, that is, regardless of whether markets are assumed to be cleared in each short period, or over longer periods, or whether no clearance but merely a tendency toward clearance is assumed.

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STOCK AND FLOW ANALYSIS: FURTHER COMMENT

By Lawrence R. Klein

Fellner and Somers have commented on my short article[1] in such a way that I must stress certain observations.

They claim that I find no loanable funds theory in the dynamic case. This is not entirely correct. I have followed some routine procedures in setting up dynamic models and can adapt them to contain either a liquidity preference or a loanable funds theory of interest. My purpose is to show that the two theories are different in a dynamic framework, and I do not rule out either one in advance. Ultimately the choice between the two theories will have to be based on empirical information.

The concept of dynamics proposed by Fellner and Somers is thoroughly unacceptable. Their definitions lead to results that are formally equivalent to those obtained in static analysis, and no number of references to distinguished persons in the field of interest theory will make their brand of dynamics any more interesting or useful. Actually, I have tried to give a more realistic picture of economic processes that yields much richer solutions. I am greatly disappointed at the insistence of Fellner and Somers on what I choose to call a "sham dynamics."

I must point out that by assuming k_t to be an exogenous variable, I am under no compulsion to assume $k_t = k_{t-1}$. This point is made only in the interest of precision since the assumption of Fellner and Somers ($k_t = k_{t-1}$), although unwarranted, plays no essential role in the treatment of the problem at hand.

The dogmatic assertion of Fellner and Somers that " . . . in *any* system, these factors [other than the demand and supply of securities] can affect the market rate of interest only through their effect on the demand and supply of interest-bearing securities" closes the door to scientific discussion. This is, self-evidently, the central assumption of the loanable funds theory, but it cannot be used to prove that the liquidity preference theory is identical, for in the latter theory an entirely different assumption is used, namely, that interest is the reward for being illiquid. Thus, the grin has a cat after all.

I close with a rhetorical question. What are the structural characteristics of their suggested unique relationship between the excess demand for money and the excess demand for claims? Is it a behavior equation for some group in the economy; or is it a market clearing equation; or is it an institutional equation; or what is it?

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[1] See the two preceding articles in this issue.

STOCK AND FLOW ANALYSIS: DISCUSSION

By Karl Brunner

The discussion between Klein and Fellner and Somers[1] about the structure of the mechanism determining the rate of interest has the merit of focusing attention on two central issues: In the first place, we have to choose between a flow or a stock theory of interest. Klein maintains that these two theories differ essentially and that we have to decide in favor of a stock theory. And secondly, we must decide within the stock theory between a liquidity theory and what might be called a securities theory. Again these two theories are not equivalent. An interesting result of the discussion is that it calls attention to the different types of behavior in the securities market implied by the two theories—a fact reflected in the dynamic adjustment equation.

ISSUE: STOCK OR FLOW THEORY

1. Fellner and Somers maintain that, by analyzing observable phenomena as if they were the solutions of a succession of static equational systems, the flow and the stock theory of interest coincide. This is correct, and was found to be a useful formulation in earlier articles by Fellner and Somers where the main concern was the relation between the loanable funds theory and the liquidity theory within a static framework. It is definitely less useful, however, when the relevancy of distinguishing between stock and flow variables in economic analysis is the point at issue. Within a static framework, as presented by Fellner and Somers, no obvious choice is possible between a flow and a stock theory. In fact, the problem will only be obscured by assuming away the adjustment processes. Stock and flow theory will differ essentially as soon as observable phenomena are treated as solutions of dynamic equational systems describing the structure of adjustment processes.

2. Klein's approach has the advantage of linking up interest theory with a general theory of price determination of goods, with large stocks and relatively small changes in net supply and demand where the decision to hold the stock is continuously appraised in the light of current market situations. Such a situation describes, surely, the essential features of the capital market. Under these circumstances the momentary price will be determined by the stock relationship and not by any flow relationship. In general, a flow equilibrium price will still be accompanied by a stock disequilibrium, which will force a change in price till short-run stock equilibrium has been achieved. This situation will then be ac-

[1] This is the fourth of a series of five notes on this general topic, all appearing in this issue.

companied by a flow disequilibrium which will change the underlying stock relationship. So a new stock equilibrium will be formed, with a new price and a new flow disequilibrium, and so on until the whole short-run system has eventually been adjusted.

3. A simple mathematical model of such a situation would be as follows:

$$S(t) = \int_{t-1}^{t} s(\tau)\, d\tau + k, \tag{1}$$

$$s(t) = s[p(t - \theta)], \theta \geqq 0, \tag{2}$$

$$S(t) = D[p(t)], \tag{3}$$

where $S(t)$ = accumulated stocks at time t, $s(t)$ = rate of change of S at time t, p = price of a commodity unit, and D = quantity demanded.

The flow equilibrium price is defined by the relation

$$s[p(t - \theta)] = (d/dt)D[p(t)], \tag{4}$$

whereas in (3) a stock relation price is determined. These two prices will coincide only for the stationary solution of the dynamic system presented.

4. Looking over the system we can distinguish two situations: (a) full equilibrium, implying both stock and flow equilibrium, and (b) partial equilibrium, implying (i) either stock equilibrium and flow disequilibrium or (ii) stock disequilibrium and flow equilibrium. (a) describes the stationary solution—or the static counterpart— of a dynamic theory; and (b) reflects the nonstationary solution of a dynamic theory. In (b) the momentary price is clearly determined by the stock relation, whereas in (a) it looks at first as if either the stock or flow relation could do the job. In fact both are necessary; the price is determined by the equilibrium of stocks under a special restriction; in the case at hand this restriction is that flows be zero. Within short-run static systems the special restriction is formed by the equalization of flows. In both (a) and (b) the stock relation enters the pricing mechanism in some essential way.

5. A real flow theory ought to postulate that flows dominate the scene, that is, that the phenomena to be explained are to be conceived essentially as solutions of flow relations, whatever the stock relation may be. This is not Fellner and Somers' position. They first assume stock equilibrium (as was done by Keynes and Lerner, with whom they were concerned) and then show the relation between stock and flow equality. In symbols: what Klein asked for was a choice between the following two hypotheses:

$$r_t - r_{t-1} = G(B_t^S - B_{t-1}^S - B_t^D + B_{t-1}^D), \tag{5}$$

with the implied restriction, $B^S_{t-1} - B^D_{t-1} \neq 0$, and

$$(6) \qquad r_t - r_{t-1} = G(B^S_t - B^D_t).$$

Fellner and Somers answer by reiterating their restriction, which is the contrary of Klein's tacit restriction. But then the result is not equivalence between a real flow and a real stock theory. If stock equilibrium in $(t - 1)$ is a necessary condition for the proposition that flow equality implies and is implied by stock equality in t, then surely the stock relation has to form an integral part of the theory. The restriction imposed by Fellner and Somers obscures an important question: what the general structure of economic processes is when demand and supply are not simultaneously equal for both stocks and flows.

In terms of our simple model, Klein's question would be as to whether the momentary price is determined by equation (4) or by equation (3). And here no equivalence is to be found in general. In the nonstationary case, (3) alone determines the momentary price, and in the stationary case (3) and (4) together determine price and a stock corresponding to this price, so that the price may actually hold.

ISSUE: LIQUIDITY THEORY OR CREDIT MARKET THEORY

6. The first issue of the discussion between Klein, on the one hand, and Fellner and Somers, on the other, was concerned with the relation of stocks and flows in the theory of price determination of goods with relatively large stocks and small net supplies and net demands. The second issue is guided by the question as to which stocks are the relevant ones in the mechanism determining interest. Whereas the first issue appears to be settled in favor of what may be termed a stock theory (which does not imply that the flow relation is irrelevant), the second issue does not seem to be settled yet. Two hypotheses disputed are the liquidity theory:

$$(7) \qquad r_t - r_{t-1} = F(k_t - M^D_{1t} - M^D_{2t}),$$

and the securities theory:

$$(8) \qquad r_t - r_{t-1} = G(B^S_t - B^D_t).$$

It is interesting to observe that these two hypotheses have different implications with respect to the structure of market mechanisms. Ultimately they will have to be judged by this fact.

7. The securities theory implies the traditional notion of market behavior, well described by Fellner and Somers: "... In a multidimensional system there are a great many factors which affect the interest rate. However, in *any* system, these factors can affect the market rate of interest only through their effect on the demand and supply of in-

terest-bearing securities." The market behavior implied is described by the well-known dynamic market-system formulated by Lange[4] and Samuelson,

$$(dp_i/dt) = H^i(X_i^D - X_i^S) \qquad (i = 1, \cdots, n), \tag{9}$$

where, denoting by X the quantity of a commodity, we have

$$X_i^D = X_i^D(p_1, \cdots, p_n); \qquad X_i^S = X_i^S(p_1, \cdots, p_n). \tag{10}$$

Equation (2) reflects this type of behavior.

The liquidity theory implies a distinctly different behavior on the securities market. This can be seen in the following way: The liquidity theory is based on the hypothesis expressed by equation (9a) of the Klein paper. By various substitutions we can then derive equation (12.1) of the paper by Fellner and Somers.

The adjustment equation of the securities market is thus of the general form

$$\frac{dp}{dt} = F\left(X_i^D - X_i^S, \frac{dp_1}{dt}, \cdots, \frac{dp_{i-1}}{dt}, \frac{dp_{i+1}}{dt}, \cdots, \frac{dp_n}{dt}\right). \tag{11}$$

Such a mechanism implies that the rate of interest may move even when the credit market is balanced, because of disequilibria in other parts of the system.

8. By comparing (9) and (11) we see clearly the difference in market behavior implied by the liquidity theory and the securities theory. If we accept (7) as a valid presentation of the liquidity theory of interest—and on the surface it looks very plausible—we are bound to acknowledge a market behavior in the securities market as described in (11). This implication and the resulting difference in comparison with the securities theory seems to have been overlooked in the past, and to have it clearly focused is a merit of the discussion between Fellner and Somers and Klein. We may point out that the stationary solutions of (9) and (11) as concerns r will coincide. Thus, for any discussion keeping rigidly to a static framework, there is no need to distinguish between a securities and a liquidity theory of interest. The choice between the two hypotheses may show up in the purely formal question as to which one of the n dependent equations in $(n - 1)$ variables to delete. But an unambiguous definition of the two hypotheses cannot be presented in purely static terms.

When we look at (9) and (11) we have no difficulty in regarding (9) as a plausible type of adjustment behavior. But (11), on the other hand, presents a troublesome conception of market behavior. It strains our

[4] Oscar Lange, *Price Flexibility and Employment*, Cowles Commission Monograph No. 8, Bloomington, Ind.: The Principia Press, 1944, Appendix, p. 91.

imagination to think of a change in price of a specific commodity because of disequilibria in other markets of the system, without the necessity of first breaking up the market equilibrium of the commodity under consideration. Actually, there exist institutional set-ups which imply such a process—but have they been observed in securities markets? In evaluating (11) we have to remember that the dynamic variables included came in because of substituting excess supply of money (in terms of stocks) by the sum of the excess demand for bonds, for commodities, and for labor. By inverting the last two excess demand functions we have rates of change of prices included. Thus the dynamic variables in (11) are independent of a generalized Keynesian hypothesis with respect to the liquidity function, an explicit formulation of which would be:

$$M^D = L[r_t, \phi(r_{t-1}, \cdots, r_{t-n}), p_t, \psi(p_{t-1}, \cdots, p_{t-n}), Y], \tag{12}$$

where r = rate of interest at various points of time, p = prices, and Y = money income. The ϕ and ψ functions embody the accumulation of past experiences with respect to securities' prices and other prices. They shape the notion of "normal" price and interest. The relation between this quantity and the actual price was conceived as crucial for the demand for money. On such a basis, X^D and X^S in (11) could be conceived as limited to the static part of (12), while mentioning the dynamic part contained in ϕ and ψ separately. But this interpretation assumes that the excess supply of money and the excess demand for bonds are identically equal. And so (11) and (9) would actually be equivalent. X^D and X^S in (9) would then differ from X^D and X^S in (11) by including the dynamic part of L, which is transmitted to the bond equations via the interdependence provided by the budget restriction. In this way, unfortunately, the "Keynesian hypothesis" does not help in rationalizing the dynamic features of (11). It may well be that Klein has a good rationalization in the back of his mind for setting up his equations (9c), (10c), (11c), but this economic rationale has not been convincingly or sufficiently developed. In this respect, as the discussion stands, the balance is definitely tilted in favor of Fellner and Somers, but Klein is correct in maintaining that the two stock theories are different.

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STOCK AND FLOW ANALYSIS: NOTE ON THE DISCUSSION

By William Fellner and Harold M. Somers

Thanks to the preceding discussion, certain differences of opinion seem to be almost completely bridged. We all seem to agree that the "liquidity preference theory" is different in essential respects from the loanable funds theory *only if* the former is understood to imply that the *rate of interest may move even though the demand for claims equals the supply of claims at the prevailing interest rate.* The one difference which remains is merely that the present writers reject this as an acceptable interpretation of the Keynesian "liquidity preference theory" because in the Keynesian liquidity preference analysis, just as in the loanable funds theory, the rate of interest is defined as the price of claims (*General Theory*, p. 222) and is *strictly distinguished from the marginal efficiency of capital.*

We do not feel that an approach should be termed "static" merely because it makes the assumption that some market is cleared during a definable period of time. But it is quite possible that Klein's type of dynamic approach, which avoids this assumption, will prove more useful in economic analysis. All participants in the present controversy seem to agree that the equivalence of the loanable funds theory with the liquidity preference theory is not affected by the mere decision to adopt Klein's concept of dynamics (although, in his kind of dynamic approach, stock analysis ceases to be equivalent to flow analysis). In order to reach the conclusion that the loanable funds theory is *not* equivalent to the liquidity preference theory, it is necessary to interpret the "liquidity preference theory" in a way which we consider to be unacceptable for the reasons given in the first paragraph of this note.

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Editor's Note—The above statement was prepared by the authors before Dr. Klein's "Further Comment" was available. Publication arrangements also prevented the inclusion of a reply by Dr. Klein to the discussion by Dr. Brunner.

RISK ALLOWANCES FOR PRICE EXPECTATIONS

By Paul B. Simpson

Entrepreneurs may form their plans on the basis of expected prices equal to most probable prices plus or minus risk allowances. The allowance may be interpreted as a certainty equivalent of subjective probability calculations, and may be given exact formulation under some conditions. The risk allowance will be positive whenever entrepreneurs are willing to sacrifice expected returns for a narrowing of dispersion of possible returns. Diminishing marginal utility of money will explain such conservatism. Limited capital resources also tend to make entrepreneurs behave in a conservative manner.

SUMMARY

An entrepreneur who seeks to maximize his profits over time must plan a stream of expenditures for productive factors and a stream of receipts from sales. He will be forced to form estimates of the prices that he must pay and that he may expect to obtain for his products. If he forms such estimates, making suitable allowances for interest charges, and if he has available the necessary technological information regarding production possibilities, he can choose a production plan which will maximize his expected profit. Such a maximized profit plan implies that the ratio of marginal substitutabilities of each two inputs or outputs is equal to the ratio of expected prices.

The entrepreneur can never be certain of his expectations, however, and this fact will modify his decisions. Some writers have suggested that an entrepreneur will add risk allowances to his expected prices as a kind of "insurance." An alternative procedure is to consider the rational behavior of an entrepreneur who forms expectations in terms of subjective probabilities. It turns out that risk allowances can be interpreted in terms of such probability formulations. The ratio of marginal substitutabilities of each two units of inputs or outputs will equal the ratio of expected prices, plus or minus risk allowances expressed in terms of the dispersions of the expected prices and of the entrepreneur's preferences for dispersion in his anticipated profits. If the entrepreneur has a preference for more certainty, that is, for smaller dispersion of gains, then the risk allowances will be positive, and factors and products will be avoided which have a wider range of probable prices. The converse is true if the entrepreneur likes the possibility of widely dispersed results of his decision.

Aversion to dispersion can be interpreted in terms of utility analysis. If it is assumed that the entrepreneur places an evaluation or utility upon different profits, then it may be presumed that the entrepreneur

wishes to maximize his expected utility, not his expected profit. The principle of diminishing marginal utility of profit implies that more dispersed expectations of profits are not desired.

An entrepreneur will prefer relatively certain situations because his capital resources are limited. Business is handicapped in situations of uncertainty because it has limited capital and for that reason cannot count upon its luck balancing out over a period of time. Hence conservative practices are likely to be adopted.

* * * *

Professor J. R. Hicks [3] has introduced the notion of expected prices equal to most probable future prices plus or minus allowances for risk into his formulation of entrepreneurial planning. Extensive use of the device has been made by Professor Oscar Lange [4]. It is interesting to interpret these adjusted price expectations in terms of the theory of subjective risk [5, 7]. This can be done very simply under the following special assumptions:

I. An entrepreneur forms for planning purposes estimates of future prices for all commodities that affect him. A separate commodity is defined for each time period in the future (of which there are a finite number) and for each commodity with different substitutionary characteristics within a given time period. These estimated prices p_i, $(i = 1, 2, \cdots, n)$, are random variables with means $\mathcal{E}p_i = \bar{p}_i$, and covariances and variances $\mathcal{E}(p_i - \bar{p}_i)(p_j - \bar{p}_j) = \sigma_{ij}$, where $\sigma_{ii} = \sigma_i^2$. These prices may incorporate an appropriate interest discount or premium.

II. The entrepreneur will plan to buy or sell quantities q_i, where positive q_i denote sales (products) and negative q_i denote purchases (factors). The sensible alternatives of sets (q_i), among which choice must be made, are given by a production function $Q(q_1, q_2, \cdots, q_n) = 0$ which has first derivatives for the relevant ranges of the variables.

III. The calculated profit $V = \Sigma p_i q_i$ is seen by the entrepreneur as a normally distributed random variable with mean $\bar{V} = \Sigma q_i \bar{p}_i$ and variance $S^2 = \Sigma_{i,j} q_i q_j \sigma_{ij}$. The two parameters $\bar{V}$ and S give the information about each possible plan which the entrepreneur considers in making his choice of plan. It may be noted that V will tend to be normal regardless of the shape of the distributions of the p_i unless the prices are highly correlated, or unless the number of prices is very small.

IV. An analytic function $\phi(\bar{V}, S)$ exists for each entrepreneur, ordering the preferences of the entrepreneur for assigned values of $\bar{V}$ and S. The larger ϕ, the more preferred the position. If S is constant, ϕ will increase with $\bar{V}$. It may be presumed that the capital resources of the entrepreneur are limited and fixed, since ϕ may well depend on this parameter as well as on subjective factors, gambling instinct, etc.

V. Each p_i is independent of q_j, ($j = 1, 2, \cdots, n$), that is to say the actions of the entrepreneur have a very small influence on prices. The entrepreneur does not act as a monopolist or monopsonist.

A solution of the problem of determining the maximum position for the entrepreneur may be sought by the familiar method of maximizing ϕ with the side relation Q. A solution will be given in some cases by the necessary conditions:

$$\frac{\dfrac{\partial Q}{\partial q_i}}{\dfrac{\partial Q}{\partial q_j}} = \frac{\dfrac{\partial \phi}{\partial q_i}}{\dfrac{\partial \phi}{\partial q_j}} = \frac{\dfrac{\partial \phi}{\partial \bar{V}}\dfrac{\partial \bar{V}}{\partial q_i} + \dfrac{\partial \phi}{\partial S}\dfrac{\partial S}{\partial q_i}}{\dfrac{\partial \phi}{\partial \bar{V}}\dfrac{\partial \bar{V}}{\partial q_j} + \dfrac{\partial \phi}{\partial S}\dfrac{\partial S}{\partial q_j}}$$

$$(1) \qquad = \frac{\dfrac{\partial \phi}{\partial \bar{V}}\bar{p}_i + \left(\dfrac{\partial \phi}{\partial S}\right)\left(\dfrac{1}{S}\right)(\sum_j q_j \sigma_{ij})}{\dfrac{\partial \phi}{\partial \bar{V}}\bar{p}_j + \left(\dfrac{\partial \phi}{\partial S}\right)\left(\dfrac{1}{S}\right)(\sum_i q_i \sigma_{ij})}$$

where ($i, j = 1, 2, \cdots, n$). The equations (1) together with the identities of II and III give $n + 2$ equations with which to solve for $n + 2$ unknowns. Sufficient conditions are complex and will not be given.

Increasing returns to scale will generally preclude a solution of the type given by (1), although not necessarily so since the risk preference function may be such as to offset the influence of increasing returns. Monopolistic conditions would tend toward a solution also, as in the case of the traditional theory of the firm.

It is convenient to rewrite (1) introducing the slope of the indifference curve at the point of equilibrium:

$$(2) \qquad \frac{\dfrac{\partial Q}{\partial q_i}}{\dfrac{\partial Q}{\partial q_j}} = \frac{\bar{p}_i - \left(\dfrac{1}{S}\right)\left(\dfrac{d\bar{V}}{dS}\right)(\sum_j q_j \sigma_{ij})}{\bar{p}_j - \left(\dfrac{1}{S}\right)\left(\dfrac{d\bar{V}}{dS}\right)(\sum_i q_i \sigma_{ij})}.$$

The usual condition for the equilibrium of the firm is that the entrepreneur selects a production plan such that the ratios of marginal substitutabilities equal the ratios of prices for each two commodities. Equations (2) express similar conditions, but in place of prices we have expected prices, $\bar{p}_i$ and $\bar{p}_j$, plus or minus expressions which are equivalent to Hicks's risk allowances. The risk allowance can be considered as a certainty equivalent of choice among probability alternatives.

Hicks has assumed that the risk allowance is positive for purchased goods and negative for products to be sold. This assumption will be justified in the following case: Assume that net correlation among prices,

i.e., the sum of terms involving σ_{ij}, $i \neq j$, is small compared to σ_{ii}, and that $d\bar{V}/dS > 0$. Under these conditions the effective price given by the equilibrium ratios (2) is higher than $\bar{p}_i$ for factors (q_i negative) and lower than $\bar{p}_i$ for products (q_i positive). Commodities whose prices have small dispersion will be preferred under these conditions, other things being equal.

The condition $d\bar{V}/dS > 0$ implies that the indifference curve for $\bar{V}$ and S when ϕ is constant is increasing. This means that a wider dispersion of results is acceptable only if a higher expected return is obtainable. This would seem to be a conservative practice since wider dispersion of results would not be avoided perhaps by a more audaciously inclined person. Thus Hicks's formulation implies conservatism in the sense $d\bar{V}/dS > 0$ at the point of equilibrium.

Conservatism in this sense can be interpreted in terms of utility. Utility notions which have not been popular with economists in recent years have been shown by von Neumann and Morgenstern [6] to have important advantages in treating choices among chance alternatives. The two formulations of choice, that of utility and that of preference functions, are closely related. A probability density function of a random variable V can be defined in a wide variety of cases by a function of finite number of moments. Let such a density function be $\varphi(V, m_1, \cdots, m_k)$. Let a preference function of the moments be $\phi(m_1, \cdots, m_k)$. If there exists a function $U(V)$ such that

$$\phi(m_1, \cdots, m_k) = \int_{-\infty}^{+\infty} U(V)\varphi(V, m_1, \cdots, m_k)\, dV, \tag{3}$$

then U and ϕ are equivalent as regards choice among the probability density functions φ. Any choice of production plan which maximizes ϕ will also maximize the expected utility. In general there will not exist a solution $U(V)$ for the integral equation (3) for any ϕ and φ. Hence it is not possible to find a utility equivalent of every preference function. On the other hand, it will be possible in general to find a preference function corresponding to a utility function $U(V)$ by integrating the right-hand side of (3). Thus the types of decisions which can be formulated by utility functions form a subclass of the types definable by preference functions. However some types of preference functions can be translated into utility functions and for such types further statements regarding conservatism are possible.

Assume that the utility of a gain, V, for an individual is given by a utility function $U(V)$ which is analytic almost everywhere for all real V. It may be assumed that $dU/dV > 0$ for all V for which the derivative exists, since individuals generally prefer more to less. However it might be that the derivative is zero in certain ranges of V, especially for very

high and very low ranges of V. Diminishing marginal utility exists where $d^2U/dV^2 < 0$.

In utility analysis it is assumed that an individual arrives at a decision by maximizing his expected utility. For normal distribution the expected utility is

$$\bar{U} = \int_{-\infty}^{+\infty} U(V)N(V, \bar{V}, S)\, dV, \tag{4}$$

where $N(V, \bar{V}, S)$ is the normal density function. It will be assumed that the integral exists.

We are interested in the effect of a change in S for a given $\bar{V}$. Letting $x = (V - \bar{V})/S$ and differentiating the resulting integral with respect to S, we obtain

$$\frac{\partial \bar{U}}{\partial S} = \int_{-\infty}^{+\infty} \frac{dU}{dV}(\bar{V} + Sx)\, x\, N(x, 0, 1)\, dx. \tag{5}$$

If for all positive x,

$$\frac{dU}{dV}(\bar{V} - Sx) \geqslant \frac{dU}{dV}(\bar{V} + Sx), \tag{6}$$

and for a set of x of nonzero measure the inequality holds, then $\partial \bar{U}/\partial S < 0$. Thus a utility curve which is convex from above almost everywhere implies that dispersion is not desired. Of course (5) may be negative when only a part of the utility curve is convex, but no general statement is possible in this case. A similar result has been stated by Friedman and Savage [2, p. 291].

It has been pointed out by Professor Fellner [1] and others that an entrepreneur may act conservatively because he has limited capital resources. In making a decision an entrepreneur is likely to use principles that will promise desirable results if repeated in similar cases. The decision process is then a rule of life to be used over and over. When repeated decisions are considered, the importance of expected or average returns seems to be enhanced, and the importance of dispersion decreased, since the magnitude of the dispersion involved would be reduced as the number of decisions increases. The entrepreneur could count on obtaining near-average results. This would be the case except for the fact that the entrepreneur has limited capital. With limited capital an initial loss handicaps or destroys the opportunity to engage in further entrepreneurial activities. An entrepreneur is in a situation similar to that of the gambler of classical probability theory who plays a game repeatedly for stakes against an opponent of infinite capital [8]. Given sufficient capital, a gambler can count upon his luck averaging out. With inferior capital

a gambler may be ruined by a streak of bad luck, and the latter possibility is not at all unlikely. Similarly an entrepreneur in entering into competition with the business world faces an opponent of vastly superior capital. Thus the dispersion of results in the individual decision is of greater importance than otherwise would be the case. Repeated plays do not lower the importance of dispersion, but may in fact enhance it. A risky decision involving more dispersed results will be avoided in an individual case. Also the amount of capital ventured is likely to be restricted and dispersion of results reduced in this manner.

As an illustration of the effects of limited capital on dispersion, consider the following simple example. Suppose a gambler plays a game where the probability of obtaining -1 dollar for each dollar staked is α, the probability of obtaining 0 is $(1 - 2\alpha)$, and the probability of obtaining $+1$ for each dollar staked is α. The gambler has an initial fortune F. He bets a certain portion, r, of the fortune that he possesses at the time of play. The expected fortune after one play is F and the variance is $F^2\, 2\alpha r^2$. If the entrepreneur plays the same game G times, staking an rth of his existing fortune each time, his expected result is F and the variance is

$$F^2\{(1 + 2\alpha r^2)^G - 1\} = F^2\left\{G2\alpha r^2 + \frac{G(G-1)}{2}(2\alpha r^2)^2 + \cdots\right\}.$$

The effect of repeated plays is not to reduce the dispersion as compared with one play, but to increase it. Very high possible returns for the series of games are balanced by a large probability of loss of part of the initial fortune. The larger the dispersion of the individual game, as measured by α and r, the larger is this spread in dispersion of results of repeated play. Under such conditions a gambler finds no "protection" in repeated plays. In fact the limitations of his capital place him at a disadvantage in the sense that losses are highly probable. It may be, of course, that the small possibility of large gain is still attractive to the entrepreneur. The point is that limited capital tends to magnify the importance of dispersion.

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A NOTE ON BRAND-NAMES AND RELATED ECONOMIC PHENOMENA

By George Kingsley Zipf

A number of observed cases are reported of doubly logarithmic rectilinearity with least square fits (1) in the phenomenon of brand-names (including trade-names) of manufacturers and distributors in the United States (1946) in reference to (a) the varying number of concerns that use the same brand-name, (b) the varying number of brand-names in cities of origin of different population sizes, (c) the varying preferences of families for different brands of like commodities, (d) the varying number of different items under like brands; and (2) in the closely related phenomenon of (a) the varying number of different manufacturers of like goods, (b) the varying number of factories in cities of different population size, and (c) the varying number of manufacturers with different numbers of warehouses and other subsidiary organizations.

The impetus to the present investigation arose from the reflection that the frequency distribution of brand-names (including trade-names) might be similar to that observed for words[1] in sizable samples of connected speech in which the x-number of different words of like y-frequency of occurrence followed with close approximation the equation $y^2 = C/x$.

The subsequent observation that the y-number of different firms using the same x-number of brand-names was distributed according to the foregoing equation led to a study both of the historical background and of the legal aspects of the proprietary rights to brand-names in the United States under the Lanham Law (in effect since July 5, 1947).[2]

Under the restrictions of this law there are three chief factors governing the proprietary right to brand-names that are designed primarily to avoid deception or confusion as to the origin of the goods of commerce: (1) the *territorial* factor, to the effect that like products competing in the same territory may not have the same brand-name; (2) the *product-classification* factor, to the effect that like brand-names may be used by different owners only for distinguishably different kinds of products; and (3) the *linguistic dissimilarity* factor, to the effect that brand-names for competing products must be sufficiently dissimilar linguistically to avoid confusion in the buyer's mind.

Because of the importance of these three factors, the present investi-

[1] G. K. Zipf, *Human Behavior and the Principle of Least Effort*, Cambridge, Mass.: Addison-Wesley Press, 1949, 564 pp.

[2] H. D. Nims, *The Law of Unfair Competition and Trade-Marks*, Third Edition, New York: Baker, Voorhis and Co., 1929. For penetrating discussion of the Lanham Act, cf. Neil H. Borden, "The New Trade-Mark Law," *Harvard Business Review*, Vol. 25, Spring, 1947, pp. 289–305.

gation of brand-names included studies (1) of the number of competitive manufacturers of like goods; (2) of the distribution of factories, branch-offices, and other subsidiary organizations throughout the country; and (3) the distribution of brands in cities, and the consumer preferences for brands within cities, as well as related problems.

The information used in this investigation was contained in (1) *Thomas' Register of American Manufacturers*[3] for 1947 which includes among other things: manufacturers classified by products; manufacturers in alphabetic order with home offices, branches, location of factories, etc.; trade-names (brand-names) classified by owners with addresses; index to products; (2) *The Chicago Sun-Times Pantry Poll*[4] which reports the inventories of branded groceries in the pantries of 400 control families throughout the greater Chicago shopping district; (3) the *Illinois Daily Newspaper Markets First Annual Consumer Analysis* (July–August, 1946)[5] which reports the different brands of an assortment of consumers' goods stocked and purchased in 44 Illinois cities of varying size outside Chicago.

The technique employed was that of fitting, by least squares, lines of regression upon doubly logarithmic coordinates with the calculation of root-mean-square deviations in each case.[6]

In the summary which follows, the regression equation and the error in the regression coefficient are indicated in brackets for each case.

1. The x-number of different brand-names in the United States (entire population in *Thomas' Register*) used by the same y-number of firms is approximately inversely proportional to y^2. [$\log y = -0.4711 \log x - 1.890$; ± 0.1587]

2. The x-number of different manufacturers (entire population in *Thomas' Register*) that makes the same y-number of products is approximately inversely proportional to y^2. [$\log y = -0.5080 \log x + 2.5292$; ± 0.0548]

3. The y-number of different factories (entire population in *Thomas' Register*) in cities of x-population size (1940 Census) in class intervals of 5,000 persons up through 100,000, and thence in class intervals of

[3] *Thomas' Register of American Manufacturers, 1947*, Vols. 1–4, 37th Edition, New York: Thomas Publishing Co., December, 1946.

[4] *The Chicago Sun-Times Pantry Poll*, Chicago: The Chicago Sun-Times, No. 9, April, 1948, pp. 6–15, 18–26 through "food specialties."

[5] *The Illinois Daily Newspaper Markets First Annual Consumer Analysis, July–August, 1946*, Springfield, Ill.: Illinois Research and Survey, 1946.

[6] In the initial classification and tabulation of the original data in *Thomas' Register* for the observations reported below in Cases 1, 2, 4, 5, and 6, I was materially helped by the Harvard and Radcliffe students in my course given at Harvard University during the Spring Term, 1949, and by my assistant, Mr. John P. Boland, Jr., who aided in the curve fitting.

10,000 up through 500,000, and thence in absolute population sizes, was approximately in direct proportion to x. [$\log y = 0.9034 \log x - 1.6189$; ± 0.2871]

4. The x-number of different manufacturers (entire population in *Thomas' Register*) that have the same y-number of different branch offices, including the almost negligible cases of warehouses, was approximately inversely proportional to y^2. [$\log y = -0.4331 \log x + 1.7911$; ± 0.1118]

5. The x-number of different manufacturers that have the same y-number of suborganizations of all kinds *except* factories (i.e., branch offices, warehouses, "district offices," "district sales offices," "sales representatives") is approximately inversely proportional to y^2. [$\log y = -0.4545 \log x + 1.9077$; ± 0.1600]

6. The y-number of different branches, including branch offices, in cities of x-population size (1940 Census) in class intervals as in paragraph three above is approximately proportional to the 1.50 power of x, [$\log y = 1.4805 \log x - 2.4449$; ± 0.2166], thereby showing a systematic exponential preference for cities of larger population.

7. The y-number of brand-names of firms located in cities of x-population size (1940 Census) in class intervals as in paragraph three above was approximately in direct proportion to x. [$\log y = 1.1588 \log x - 1.4602$; ± 0.2950]

8. The y-number of different brands of a combined list of different kinds of consumers' goods (i.e., lipstick, face powder, facial cream, rouge, deodorants, hand and face lotion, potato chips, macaroni and spaghetti, baking powder, salt, floor wax, peanut butter, laundry starch, and soap in flake, chip, powder, and granule form) in 44 different Illinois cities outside Chicago of varying x-population size varies approximately with the square root of x, [$\log y = 0.4123 \log x + 2.1292$; ± 0.1670], according to the *Illinois Daily Newspaper Markets First Annual Consumer Analysis*.

9. The y-percentage of the population in a sample of 400 families in Chicago (the *Chicago Sun-Times Pantry Poll*) that use the same x-number of different branded goods of all kinds is approximately inversely proportional to y^2. [$\log y = -0.5677 \log x + 1.6464$; ± 0.0995]

10. According to the same pantry poll, the x-number of different brand-names that label the same y-number of different products is inversely proportional to approximately y^2. [$\log y = -0.5022 \log x + 1.2761$; ± 0.1281]

Limitations of space preclude a discussion of the theoretically expected values suggested in connection with the 10 cases reported above, although the values can be deduced for the most part from a previous theoretical treatment[7] and will be made more explicit in a future one.

[7] G. K. Zipf, *Human Behavior*, *op. cit.*, Chapter 9.

The values of t (from Fisher's table[8]) for the differences between the theoretical and observed slopes are given in the table. Except for Cases 3 and 7, the deviations may be viewed as nonsignificant according to conventional usage. The observations, we remember, refer to the still "abnormal" economic conditions of the immediate postwar year, 1946, when reconversion to peacetime production had not been completed.

VALUES OF t

Case	Slope (Theoretical)	Slope (Observed)	n (Degrees of Freedom)	t
1	−0 50	−0.4711	85	1.5131
2	−0 50	−0.5080	315	.0705
3	+1.00	+0.9034	58	5.1408***
4	−0.50	−0.4331	52	.7895
5	−0.50	−0 4545	66	.7104
6	+1.50	+1.4805	57	.0423
7	+1.00	+1.1588	57	2 5842*
8	+0.50	+0.4123	42	1.2930
9	−0.50	−0 5677	41	.1919
10	−0.50	−0.5022	18	.0053

* Indicates rejection at 5 % level of significance.
*** Indicates rejection at .1 % level of significance.

Harvard University

[8] R. A. Fisher, *Statistical Methods for Research Workers*, Fifth Edition, London and Edinburgh: Oliver and Boyd, 1934, Table IV.

REPORT OF THE NEW YORK MEETING

DECEMBER 27–30, 1949

THE ECONOMETRIC SOCIETY held its American winter meeting in New York City, December 27–30, 1949, in conjunction with the meetings of the Allied Social Science Associations and the American Association for the Advancement of Science.

Sessions with the American Economic Association, American Statistical Association, Institute for Mathematical Statistics, and Section K. of the American Association for the Advancement of Science were held in the Hotels Commodore, Biltmore, and Governor Clinton. Total registration exclusive of the American Association for the Advancement of Science was over 4300. The sessions of the Econometric Society were open to all its members as well as to members of the other organizations and were well attended.

The program was arranged by a committee consisting of Harold M. Somers, University of Buffalo (chairman); Armen Alchian, The RAND Corporation; Sidney S. Alexander, International Monetary Fund; Milton Friedman, The University of Chicago; J. A. Nordin, Iowa State College; David Rosenblatt, Carnegie Institute of Technology; William B. Simpson, Cowles Commission for Research in Economics (ex officio); Rutledge Vining, National Bureau of Economic Research; and C. Ashley Wright, Standard Oil Company of New Jersey. Henry H. Villard, College of the City of New York, served as representative for local arrangements.

Following is the program of the Econometric Society sessions with abstracts of the papers and discussions presented, so far as available. The program is preceded by a list in which the material is indexed by speakers in order to facilitate subsequent reference. An asterisk denotes papers or discussions reported here only by title.

STRUCTURE OF THE LABOR MARKET

John T. Dunlop of Harvard University presided at the opening session of the Econometric Society, held Tuesday morning, December 27. The following papers on the structure of the labor market were presented:

Measuring the Supply of Labor, CLARENCE D. LONG, The Johns Hopkins University.

THE EFFECTIVE supply of labor has four dimensions: labor force, quality of worker, fulltime hours, and intensity of effort. There are four theories of how labor supply behaves, all measuring it in units of men or hours: (1) The classical theory holds that it depends on real wages. Lionel Robbins reasoned that only empirical study can tell us whether it would rise or fall under rising real incomes; and accordingly Paul Douglas tried to show it would fall. (2) The Keynesian holds that short run labor supply depends not on real but on money wages and is completely elastic. (3) The additional-worker theory holds that it increases in depression. (4) A variant, using World War II as proof, holds that it increases under full employment.

This study analyzes the labor force in five countries, fulltime hours and labor quality in the United States, and labor effort in two Baltimore concerns. It concludes that: (1) The peacetime labor force in numbers of workers bears a highly stable ratio to working age population over time and is completely inelastic with respect to real or money incomes per worker in both long and short run. (I call this the first law of labor supply.) The labor force proportion falls slightly in depression, rises a bit in peacetime prosperity, and goes up a lot in some countries during war. The World War II expansion, however, was due to the draft and not to full employment, and furthermore left no permanent mark. My findings

contradict both Douglas and Keynes; the labor force over long or short periods of time is not backward sloping as Douglas undertook to show, nor elastic (horizontal) as Keynes suggested, but completely inelastic (vertical). (2) Fulltime hours in the United States since 1890 have fallen 44/100 of 1% for each rise of 1% in disposable income per hour. In the short run they have been more affected by employment; in any case, they seem only to fall. 3) Quality of worker in the United States measured in education has risen just enough to offset the decline in hours. (4) Labor effort in two Baltimore factories fluctuates widely from one month or year to another and, in the nonunion concern, tends to go up when jobs are hard to get—reducing the real cost of an hour's labor to the employer without reducing its buying power to the worker. If this happens generally, effort flexibility would tend to do the work of price flexibility in cushioning the business cycle.

An Econometric Investigation of the British Labor Market, GERHARD TINTNER, Iowa State College.

A STATIC demand and a static supply function for industrial labor in Great Britain has been estimated from data for the period 1920–1938. K is the Ministry of Labor cost of living index, P the Board of Trade wholesale price index, W an industrial wage index. D is the number of employed in industry relative to the gross labor force (unpublished data). E is the sum of the number of employed and unemployed, relative to the gross labor force. T represents time.

The method of weighted regression is used to estimate the static demand and the static supply curve of labor in Great Britain during the period. Fiducial or confidence limits are computed and the problem of multicollinearity is investigated. Finally, certain linear tests enable us to form an idea about the homogeneity properties of the demand and supply function of labor.

It appears that the demand function for industrial labor is probably homogeneous of zero degree in wages and prices (wholesale price index). The real wage elasticity of the demand for industrial labor is estimated as -0.4 without time trend and as -0.7 with time trend.

It is not at all certain that there is any dependence of the supply of labor on wages and prices. But it appears that the supply function for industrial labor is again homogeneous of zero degree in prices (cost of living index) and wages. This has a certain relevance for the Keynesian discussion. The real wage elasticity of the supply of labor is estimated as -0.2 without time trend and -0.3 with time trend. It is interesting that the elasticity is negative.

A Model of Output Restriction in the Labor Market, WALTER FIREY, University of Texas.

OUTPUT restriction by employees remains an anomaly in the theory of rational economic behavior. On-the-job studies have indicated that employees' output intentions typically fall short of management's output intentions. This discrepancy tends to persist even in the face of incentive systems, with employees consciously limiting their output to a quota which deprives them of maximum potential earnings.

In constructing a realistic model of output restriction we may posit as our basic elements: (1) the work-attitudes of employees, and (2) the work-attitudes of management. Out of the work-attitudes of employees there emerges a system of output intentions; out of the work-attitudes of management there emerges another system of output intentions. When these two systems share identical elements there is a stable alliance between employees and management, with output restriction being absent. Where the two systems have some common elements but not others there is an unstable alliance, with some output restriction being present. Where there are no common elements whatever, production ceases.

For the representation of such data the algebra of classes is well suited. Employees' output intentions may be represented as a system of sets related by inclusion, the elements of these sets being functions of employees' work-attitudes. Management's output intentions may be similarly constructed. Various modes of relationship between these two systems of output intentions may be represented in the form of set intersections having certain structural properties.

ENTREPRENEURIAL DECISIONS AND THE THEORY OF THE FIRM

The second morning session on Tuesday, December 27, was devoted to the subject of entrepreneurial decisions and the theory of the firm. Leonid Hurwicz, University of Illinois and Cowles Commission for Research in Economics, The University of Chicago, served as chairman in the absence of George J. Stigler of Columbia University. The following papers and prepared discussion were given:

The Mathematical Theory of the Firm, GABRIEL A. D. PREINREICH, New York, New York.

THE BEHAVIOR of capital goods may be analyzed by using a system of variables consisting of successive derivatives, with respect to time, of the

variable of zero order. The basic problem is to maximize the present worth of an income stream expressed in those terms up to any order n. For this purpose the calculus of variations developed by Euler only up to first order must be expanded to nth order. The resultant *rules of maximization* lead to alternative solutions, depending on how many can be simultaneously valid in a given case. *Rules for bridging discontinuities* will also emerge and add to the alternatives, of which the best available must be selected.

The familiar first-order concept *marginal income* is superseded by a flexible nth order concept called *supermarginal income*. The *proof of correct solution* is that the supermarginal income, summed up from any moment t within the jth segment up to its end, will equal the unexpired capital value of that segment alone, excluding the goodwill of all future segments, if any. This corresponds to the *law of optimum price*, a rearrangement of the broadened Euler equation to the form: "Supermarginal income always equals depreciation on capital value." The *law of replacements* is that: "A capital good should be replaced as soon as it ceases to earn interest on the excess profits of all future links in the chain, discounted to the date of replacement." All economic philosophies of the greatest benefit may be directly correlated in terms of an *index of selfishness*. The same mathematical routine thus yields any solution desired.

In the light of this analysis all extreme overgeneralizations accepted as gospel in the current *first-order theory of the firm* fall into their proper places as mere special answers to grossly oversimplified assumptions.

Inventory Theory of the Firm, Paul W. McGann, Massachusetts Institute of Technology.

This paper attempted to show how the inventory theory of the firm fits into the general theory of the firm when one analyzes the firm by maximizing a gain function subject to several conditions. The classes of conditions were listed usually as examples of how inventory "stock and flow" variables appeared in certain of these conditions under circumstances of dynamic uncertainty. The condition function types are: production functions; market functions; static multiple plant, market, and process identities; stock-flow time identities; stock maintenance functions; risk taste functions; risk transformation functions; estimation functions; and rivals' reaction functions. Entrepreneurial taste functions are maximands in sole proprietorships but otherwise are side relations conditioning the "entrepreneurial group welfare" function as maximand in other firms.

Inventory variables were found in some cases of all types of side relations except static identities and estimation functions, but they need not be left in production functions, market functions, or rivals' reaction func-

tions. Instead, risk variables such as probabilities of running out of stocks for certain periods can be used, and inventory variables would affect them through risk transformation functions. The separation of risk taste functions and entrepreneurial taste functions was tentatively justified on grounds of "introspective psychology."

The first and second order conditions of maximization were not examined. Hawtrey's inventory theory and the acceleration principle were briefly discussed and found to be special cases of the above, more general theory.

Analysis of Firm Behavior, ERNST W. SWANSON, Emory University.

ENTREPRENEURIAL decisions may become a powerful generating element and at other times a reversing element in the cumulative processes (Wicksellian). A shift to cash or near cash, given the liabilities, may be the means to a transformation from the decisions of individuals to the aggregative effect of those decisions and may be at the root of this generating element.

The extent of entrepreneurial activity will tend largely to determine what amounts of money people in general will prefer to hold. The relationship so derived is negative; as the willingness to take on risks increases, the quantity of liquid assets that people in general will be willing to hold decreases relative to income, given the change in income.

Depreciation policy may have important bearing upon decisions on assets to be held by the firm. Depreciation may be viewed as a device to prevent consumption of capital, provided that the decisions about working capital so preserved will eventuate in later replacements and provided that the depreciation debit is equal to the value of capital as dictated by current replacement prices of producers' goods. But usually depreciation accounting is on an historical basis. Therefore, during rising price levels, profits are overstated and working capital is negatively affected under the usual dividend policy. Plans to invest, if governed by profit so shown, may be overextended. Proper complementarity of the asset constellation is usually not maintained.

Assets are not homogeneous, perfectly substitutable, one item for the other. Even while there is capital consumption there may be, therefore, concurrently plans for expansion or plans for contraction, with varying multiplier effects. The multiplier thus changes with entrepreneurial decisions.

Because of asset and liability plans gone astray, the shift to liquidity in all likelihood originates with the entrepreneurs and not with households and individuals. It seems doubtful that the great majority of households finds it possible to attain to any degree of liquidity comparable to the degree attainable by firms. For the firm, liquidity seeking

appears largely as an inverse function of profit expectations and, in turn, of the willingness to assume risk.

DISCUSSION by W. W. COOPER, Carnegie Institute of Technology—The papers by Preinreich and Swanson present an interesting contrast. Preinreich seeks to precipitate a theory of the firm, and ultimately the economy, from individual asset valuation criteria. Swanson seeks to construct a theory of the firm from considerations of aggregative analysis. Preinreich's paper is theory- rather than measure-oriented, while Swanson's is measure- rather than theory-oriented. Preinreich's paper is concerned not with how businessmen do behave, but how they ought to behave. No easily discernible answer can be obtained to the test question: "Under what conditions is the suggested hypothesis false?" In Preinreich's attempt to extend the calculus of variations and to interest economists in its use, he gives inadequate attention to existence and sufficiency conditions, both of which are important to the use of this technique and may have economic significance. Swanson's paper takes on more of the character of a program than a theory. The attempt to graft class properties to the *individual* firm may not be the most fruitful approach for obtaining access to the internal structure and mechanics of the individual entities.

DISCUSSION by MERTON P. STOLTZ, Brown University. (No abstract available.)

EQUILIBRIUM THEORY AND THE DEMAND FOR MONEY

A session on equilibrium theory and the demand for money was held Tuesday afternoon, December 27. Paul A. Samuelson of the Massachusetts Institute of Technology presided as chairman and the following papers and prepared discussion were presented:

Inconsistency of Assumptions in Patinkin's Model, CECIL G. PHIPPS, University of Florida. (Published under the title, "A Note on Patinkin's 'Relative Prices,'" in ECONOMETRICA, Vol. 18, January, 1950, pp. 25–26.)

The Consistency of the Classical Theory of Money and Prices, WASSILY W. LEONTIEF, Harvard University. (Published in ECONOMETRICA, Vol. 18, January, 1950, pp. 21–24.)

The Determinacy of Absolute Prices in Classical Economic Theory, W. BRADDOCK HICKMAN, National Bureau of Economic Research. (Published in ECONOMETRICA, Vol. 18, January, 1950, pp. 9–20.)

The Rationale of Money Demand and of Money Illusion,[1] JACOB MAR-

[1] The full paper has been accepted for publication by the editor of *Metroeconomica* (Trieste) as a contribution to the volume dedicated to Professor Bresciani-Turroni.

SCHAK, Cowles Commission for Research in Economics and The University of Chicago.

CONSIDER the following propositions: (1) Each individual maximizes a utility function that depends only on the amounts of various goods he will consume during a period that begins after the marketing date and during which no further exchanges take place; (2) The *numéraire* (i.e., the thing whose price is fixed at unity) is neither a consumption nor a production good; (3) At least one individual has positive money stock. Condition (1) defines a "static" or "uniperiod" model; (2) defines paper money; (3) excludes the regime of "money-of-account."

Jointly, these three conditions imply: (4) Prices of consumption goods that would clear the (perfect) market are infinite. That is, market equilibrium is not compatible with (1), (2), (3) taken jointly. This was proved mathematically in Patinkin's first article and verbally by Phipps. It is difficult to understand why either of them could think they disagree.

Now replace (3) by: (3′) All money stocks are zero. (1), (2), (3′) jointly imply: (4′) Prices (also called absolute prices) of consumption goods are indeterminate; their ratios (so-called relative prices) are determinate and finite. Leontief's paper consists, in effect, in describing such a system: the regime of money-of-account.

Since in the system (1), (2), (3′), (4′) money stocks are zero, this system cannot be supplemented by the so-called equation of exchange making positive absolute prices proportional to the total money stock. This contradiction is present in what Hickman described as the "classical" combination of a theory of formation of relative prices (determined in the "commodity-sector" where utilities are maximized and markets cleared) with the equation of exchange (or "monetary sector"), in a regime of paper money.

To reconcile finite and determinate absolute prices of consumers' goods with positive stocks of paper money, one can drop the "static" condition (1). Replace it by: (1′) Each individual maximizes a utility function which depends not only on present but also on future consumption flows; he will be able to exchange stocks of goods and of money at future marketing dates. If some but not all individuals expect the prices of all goods to fall, these individuals will demand positive money stocks; at the same time the prices of goods will be determinate, positive, and finite.

This implies sharp alternations between an individual's bearish "flight into money" and bullish "flight into (certain) goods." More realistic, smooth fluctuations of stocks are obtained if we introduce market imperfection, including transaction costs. If barter is excluded, one can define "illiquidity" of a good as the slope of the marginal money revenue and marginal money outlay curve of the individual considered, respec-

tively, as a seller and a buyer of this good. The individual demand for stocks of various goods and of money, given expected shifts in that curve, will depend on their degrees of illiquidity: for speculative purchase to be followed by resale, a relatively "liquid" good (and money in particular) is preferred.

It is neither necessary nor sufficient to introduce uncertainty in order to explain positive stocks of money and the smooth fluctuation of these stocks.

Any system that admits positive demand for paper money stocks has also to admit that a change in the absolute prices of goods (the cheapening or appreciation of paper money in terms of goods) induces the individual to change his holdings of money relative to his holdings and consumption of goods. Hence, there is "money illusion" in all systems except that of money-of-account.

DISCUSSION by THEODORE MORGAN, University of Wisconsin—From the diverse criticisms of Patinkin's mathematical examination of the classical system, especially from Hickman's lucid paper, one concludes that the classical system "may or may not be internally consistent, may have no solution or an infinite number. The solutions, if they exist, may have negative or positive quantities." We get no comfort from the equations. And if, as Marschak argued later, it is fair to accuse the classical system of inconsistency in that no value is attributed to the *numéraire*, but only a significance in exchange, this is a simple contradiction, to be solved in simple Böhm-Bawerk terms, not needing the elaborate Patinkin analysis. But the life of the classical doctrine does not lie in logical system. It lies in the facts of western economies a century and more ago. The early economists tried to give a rough—a consciously rough—rationalization of those facts. The failure of classical economists of more recent vintage does not lie in internal inconsistency —Patinkin right or wrong—but in a failure of vision. The economic world was changing, but they repeated the doctrines of the fathers without change.

DISCUSSION by DAVID ROSENBLATT, Carnegie Institute of Technology—From a certain logico-mathematical standpoint (Weyl, Wittgenstein), the economist is confronted with three classes of languages with which to approach "economic reality": (a) Economic-model or structure-theoretic languages involving problems of model design and of mathematical and logical syntax, as well as semantic problems of representation. Patinkin's two papers as well as the methodological criticism engendered by them are restricted to this domain. (b) Measure- and instrumentation-theoretic languages involving the number fields and sets of measures defined over the economic models, e.g., index numbers, general classes of scalar and vector magnitudes. (c) Statistical-inference and hypothesis-testing languages including problems of model identification. In respect to explanation and prediction of "economic reality," where the latter phenomena are determined by measure- and instrumentation-theoretic conventions jointly with survey (micro- and macro-statistical) procedural techniques, consistency in economic model design is neither a necessary nor sufficient condition for success from the heuristic standpoint. In the face of well-defined empirical problems, formalism, whether "operationally meaningful" or not (for it need not be, cf. Schrödinger's wave function), can be flexibly molded as an instrument to the point of devising new languages, new logics, and even new metamathematics.

DISCUSSION by WILLIAM FELLNER, University of California, Berkeley, and JOSEPH A. SCHUMPETER, Harvard University. (No abstracts available.)

STATISTICAL MEASUREMENTS AND ECONOMIC ANALYSIS

The American Statistical Association joined with the Econometric Society for a late afternoon session on Tuesday, December 27, on the general topic of statistical measurements and economic analysis. W. Allen Wallis of The University of Chicago served as chairman. Papers and prepared discussion were offered as follows:

Index Number Problems in Measuring the Elasticity of Demand for Imports, ARNOLD C. HARBERGER, The Johns Hopkins University.

ECONOMISTS often lose sight of the by now well-established fact that particular index numbers generally have meaning and relevance only for particular purposes. This is especially true with respect to existing attempts at measuring the "elasticity of demand for imports," represented here by η^*.

Simple models embodying the usual assumptions of the theory of international trade, and the additional assumption (approximately valid for the imports of the United States) that cross-elasticities of demand among imported goods can be neglected, reveal that the elasticity of demand for imports is merely the value-weighted average of the elasticities of import demand for all the component imported commodities, adjusted for elasticities of foreign supply. From this result an index-number of prices P can be derived, which will be "ideal" in the sense that it will precisely estimate the true η^* from the data of any two periods, so long as demand functions have not changed in the interim.

Comparing the estimate of η^* obtained by use of the ideal index P with that obtained by using an ordinary index of prices (Laspeyres or Paasche), it can be shown that the latter estimate will be too low if, as seems likely, the prices of imported commodities with less-than-average elasticities of import demand have fluctuated more than those of commodities with greater-than-average elasticities of import demand.

To test the hypothesis that this is the case for the imports of the United States, individual elasticities of import demand were calculated for various commodities. Then those imports for which these estimates were most reliable were grouped into two categories: those with low and those with high value weights. For these two groups, and for the composite of the two, Laspeyres indices of price were constructed, and elasticities of demand for imports of the three groups were estimated using these indices. These estimates were all lower than those obtained by using the theoreti-

cally correct aggregation procedure, the significance levels of the difference being 28%, 10%, and 17%, respectively.

In interpreting these results it should be noted that the significance levels were underestimated in three ways. First, it was unfeasible to obtain adequate estimates of import demand elasticity for products in the finished-goods category, yet the presumption is strong that these goods have both high elasticities of demand (owing to high substitutability for domestic products) and relatively little price fluctuation. Second, it was not possible to obtain adequate estimates of elasticities of foreign supply: instead they were assumed to be infinite. But for all but one of the commodities considered, any deviation of foreign supply elasticity from infinity would increase the size of the difference between the two estimates compared. Finally, the significance level of the difference between the two estimates of elasticity was obtained on the assumption of zero covariance between them. Since the presumption is overwhelming that this covariance is positive, an adequate estimate of it would make the observed results more significant.

The Measurement of Extent and Growth of Monopoly in the United States, 1899–1937, G. Warren Nutter, Yale University.

Economists are continually faced with the need for estimating the comparative extent and growth of monopoly in the United States. No estimates are likely satisfactorily to meet critical standards because of serious deficiencies in data, measurement techniques, and current operational definitions of monopoly. Because of these deficiencies, only the most carefully constructed estimates can have much reliability. If disagreements about facts are to be eliminated, the issue of definition of monopoly must be considered separately from the issue of measurement of its extent. This paper is not concerned with the issue of definition. It accepts one particular set of criteria of enterprise monopoly and presents one possible index of the extent of monopoly so defined. For 1899 and 1937, "industries" are classified as monopolistic, competitive, and governmentally supervised. The degree of output concentration in an "industry" is used, with important qualifications, as the basic criterion of monopoly. For the 1890's, almost all industries taken as monopolistic are those in which the fraction of output accounted for by the four leading firms (the concentration ratio) was one-half or larger. For the 1930's, those "industries" classified by Clair Wilcox in T.N.E.C. Monograph 21 as monopolistic are taken as monopolistic. Added to these are any additional manufacturing and mining census-industries with concentration ratios of one-half or larger in 1935 and manufacturing census-products with concentration ratios of three-fourths or larger in 1937, as determined by T.N.E.C. and National Resources Committee studies of the late

1930's. Extent of monopoly (competition) is measured by the fraction of national income originating in monopolistic (competitive) "industries." Monopolistic "industries" accounted for 17% of national income in 1899 and 19% in 1937; competitive "industries," for 76% in 1899 and 56% in 1937; government and supervised "industries," for 7% in 1899 and 25% in 1937. Estimates can be broken down by industry divisions and groups. All estimates are subject to serious qualifications. First, estimates of extent as here defined are not equivalent to estimates of importance. Second, bias is introduced by the following and other factors: national income series used, relative inaccuracy of income data for the 1890's as compared with the 1930's, deficiencies in data on output concentration both time- and industry-wise, and arbitrariness of definition of monopoly. Net bias is undeterminable. The results do not, however, seem consistent with the typical case presented for the "decline of competition."

DISCUSSION by STEPHEN ENKE, The RAND Corporation—The paper discusses oligopoly rather than monopoly and seeks to describe the degree of concentration at two dates rather than any trend. The test of whether an industry is "monopolistic" or not is arbitrary, because the four firms and 50% requirements in the definition are arbitrary. Whether the conclusions are sensitive to the perhaps necessarily arbitrary definition of "monopoly" remains uncertain. Those conclusions which are presented first may be biased by the use of different criteria for the 1930's—notably the acceptance of Professor Claire Wilcox' *dicta* regarding which industries were then "monopolistic"—as compared with those employed for the 1890's. A more serious possible weakness is that industry boundaries as defined by compilers of statistics do not necessarily coincide with market boundaries as conceived by economists. It is not clear—as Nutter agrees—in what respects more or less "monopoly," as defined in his paper, would be "good" or "bad." It is not clear how or if this measure of "monopoly" could ever be used.

The Relationship of Consumption Expenditures to the Recession of 1937–1938, KENNETH D. ROOSE, University of California at Los Angeles.

THE FAILURE of consumption spending has been accorded important causal responsibility in the recession of 1937–1938. The strongest statement of what may be called the "simple" underconsumption version of the price-cost relationships in the period came from various governmental representatives. Thus, in May, 1937, Leon Henderson forecast a major business recession because prices were rising so rapidly that purchasing power was failing to keep pace. Too much of the money flow was into profits and savings where it failed to find profitable investment outlets because of the reduced consumer purchasing power. He expressed alarm because prices, as a result of monopolistic action, were rising more rapidly than wage earner incomes.

Many also attribute the imperfect recovery, 1933–1937, of certain of the investment components in the national product to the peculiar nature of that recovery. From this viewpoint, producers' durable equipment ex-

penditures expanded because investment decision was narrowly geared to consumption expenditures. But the relatively long-term commitments, such as characterized the twenties and which normally result in expansion of plant capacity, were notable by their absence. The recession, itself, resulted largely because consumption expenditures "flattened" out. The failure of consumption expenditures, operating through the acceleration principle, reacted unfavorably upon the derived producers' durable equipment expenditures and produced the phenomenal involuntary investment in inventories.

This paper examines two related propositions which arise from the considerations just raised. One is that a failure in consumption expenditures was an important cause of the 1937 downturn. The other is that most investment expenditures in the process of the recovery, 1933–1937, were *induced* by consumption expenditures. From examination of the data on consumption, it would appear that the explanation for the recession does not lie in underconsumption hypotheses. The single development among the consumption factors which may have directly affected the recession was the decline in the rate of increase in consumer installment credit. Even this decline was of slight quantitative importance. The hypotheses that consumers' goods production outstripped consumer income, and that the consumption function shifted, or that the marginal propensity to consume declined significantly appeared to be inconsistent with the data. Finally, the contention that the relative failure of long-term investment commitments and the vulnerability of the economy to recession were due to the consumption nature of the recovery has been examined and seems questionable. The principal causal explanation must be sought elsewhere, perhaps in the factors underlying private investment decision or in the cessation of net government contribution to income.

DISCUSSION by MOSES ABRAMOVITZ, Stanford University. (No abstract available.)

EXTENSIONS OF THE THEORY OF GAMES

Henry H. Villard of the College of the City of New York was chairman of an early morning session on Wednesday, December 28, which dealt with extensions of the theory of games. The following papers were presented:

A Theory of Stabilizing Business Fluctuations, LEONID HURWICZ, University of Illinois and Cowles Commission for Research in Economics, The University of Chicago.

LET x_t denote the (multidimensional) variable characterizing the state of the economy at the time t. (Thus, the real income and the price level are among the components of x.) Let $F_{t_0}(x_{t_0}, x_{t_0+1}, \cdots)$ be the (cumulative) probability distribution at time t_0 of a sequence of values $(x_{t_0}, x_{t_0+1}, \cdots)$. Assume that the community (e.g., the nation) can form a (subjective) F and that there exists a community utility function $u = \psi(F)$. (More realistic treatment would dispense with the assumption of existence of such a utility function, but would then be subject to the well-known difficulties arising in welfare economics because of the lack of interpersonal comparability. It is worth noting that this problem is often ignored in business cycle policy discussions.) In line with the customary approach one may assume that u increases with (time-discounted) average real income and decreases as the amplitude of fluctuations of x_t from one time point to another grows.

Assume that $F = \psi(c, p)$ where c represents the community's policy variable and p the public's "mood." We have $u = \varphi[\psi(c, p)] = g(c, p)$, and the policy problem is to choose an "optimal" value $\hat{c}$ of c so as to maximize u.

The writer feels that g must be regarded as a known function of c and p or the problem has no solution. It is essential, however, to consider p to be unknown. (This means that whatever is known about the public's behavior, e.g., its consumption function, has already been incorporated into g, but that there is always a "residue" of ignorance with regard to the public's behavior; the term "mood," as applied to p, is merely illustrative.)

With p unknown it is no longer possible to formulate the problem as one of simple maximization of u with regard to c. A possible approach is to require that $\hat{c}$ be so chosen as to maximize the function $h(c) \equiv \min_p \mathfrak{E}g(c, p)$, where $\mathfrak{E}$ denotes the mathematical expectation. This may be termed the "theory of games" solution though it need not imply a game interpretation of the problem. The solution $\hat{c}$ might possibly turn out to be in the nature of what in the language of the theory of games is called a "mixed (randomized) strategy." This is strongly suggested by the role of the public's expectations in the likelihood of success of a given policy c.

Complementarity and Substitution in the Theory of Games, OSKAR MORGENSTERN, Princeton University.

THE PHENOMENA of complementarity and substitution play a large role in economics. The latter is generally assumed not to produce any particular difficulties. On the contrary, whenever there is complete substitutability, this is assumed to assure identity of values of all units of all factors for which substitution holds. Complementarity, on the other hand, has offered enormous difficulty, be it in the Austrian theory of imputa-

tion or in the modern treatment of Hicks and others. This is due to the nonadditivity of value. Recently the great importance of complementarity has been denied by Samuelson. The theory of games is designed to deal fully with nonadditive value which is found to be significant in all essential games, i.e., those where there is an advantage in combining. The characteristic function, on which the theory of the n-person game is built, completely describes the cases of complementarity with which the economist is concerned. As for substitution, it is shown in the theory of simple games that far more complicated situations arise than are envisaged in economics. That theory provides examples that can be clearly identified with typical economic situations where substitutability of factors exists and yet no values at all can be assigned to the factors. This emerges when there are 6 players or factors involved. These are, therefore, counter-examples that serve to upset the above-mentioned belief that substitutability is an assurance that fully determinate equilibria of the type that economic theory is looking for can be found. Instead of pursuing that aim, it is necessary to fall back upon the quite different and far more complex notions of "solution" as developed by the theory of games thus far.

CROSS-SECTION DATA AND PROBLEMS OF AGGREGATION

Cross-section data and problems of aggregation were the subjects for the second morning session on Wednesday, December 28. The following papers and prepared discussion were presented, under the chairmanship of Gardner Ackley of the University of Michigan:

The Integration of Cross-Section and Time-Series Data, LAWRENCE R. KLEIN, National Bureau of Economic Research and Survey Research Center.

A RECOMMENDED procedure for the estimation of parameters in cross-section studies is to construct a joint distribution function of all the variables involved and obtain maximum-likelihood estimates from it. The joint distribution function can be partitioned into a product of conditional and marginal distribution functions. If the latter is independent of the parameters to be estimated, the maximum-likelihood procedure is simplified and reduces, in some cases, to the ordinary least-squares approach. In some cases where the marginal distribution does depend on the parameters to be estimated, it can be shown that least-squares estimates coincide with maximum-likelihood estimates for large samples.

In reconciling time-series estimates with cross-section estimates two fundamental points must be kept in mind. (1) As Haavelmo has stressed,

we must work with an enlarged system of explanatory variables in cross-section estimates to put all the individuals on a homogeneous basis. (2) In cases where it is not possible to account for all individual differences by an objective set of measurements and hence where the parameters to be estimated differ for each individual, it can be shown that cross-section methods and time-series methods each yield estimates of different averages of the parameters involved.

In the most general case, we should not look upon cross-section and time-series approaches as alternatives but should instead pool the two types of samples. A formal structure of the pooling approach has been worked out for linear systems with constant coefficients.

Some of these ideas have been incorporated into an empirical study of the production of railway services. In this study variables were introduced to attempt to put the individual companies on a homogeneous basis. The model estimated from cross-section data in a given year was extrapolated with industry aggregates in a surrounding year. This experiment tried to show a bridge between cross-section and time-series data and between micro- and macro-economic analysis.

The Consumption Function and the Theory of Aggregation, Melville J. Ulmer, U. S. Department of Commerce.

When individual consumption functions are added, it is found that the dollar value of total consumption is a function of the individual prices of all commodities, the level and distribution of incomes, the level and distribution of household inventories, the level and distribution of liquid assets, the rate of interest, expectations, and tastes. Most broadly conceived, the problem of aggregation consists primarily of simplifying this function so that it will be susceptible to measurement and more useful for theoretical purposes as well. For theoretical purposes, this can be done by the familiar method of holding "all other things equal." Thus aggregate consumption may be considered—precisely—a function of aggregate income alone; in much the same way as in micro-economics, the amount demanded of any product may be viewed as a function of its price alone.

But if we apply the concept of the consumption function to time series for purposes of measurement, or to problems of policy dealing with events over time, the numerous variables other than aggregate income cannot be waived away, nor can they in any extensive manner be accounted for statistically. Chief reliance must be placed on the fact that income is ordinarily the most important variable determining consumption and on the hypothesis that the other variables are either themselves closely correlated with income or, as a practical matter, that they are of negligible importance.

For this reason all statistically estimated consumption functions are, at best, rough approximations, and, as these terms have been used by Duesenberry,[1] they are all "derived" rather than "fundamental" relationships. For this reason as well we would expect the error terms of such relationships to be characterized by high positive autocorrelation, an expectation which appears to be justified by the facts.

The meaning of statistically derived consumption functions is further qualified by the problem of deflation involved in their measurement. Upon analysis it is found that the volume of consumption for the nation as a whole has meaning only in terms of a *range* of numbers within which specification is indeterminate.

Additional questions are raised concerning the usefulness of empirical consumption functions when application is made to the problem of forecasting. Specifically, there is the probability that at times consumption may vary in a significant way quite apart from changes in the level of disposable income, and since such variations would introduce changes in the same direction in disposable income, the high correlation between these variables would remain intact. Alterations in the household stocks of consumer durable goods, for example, are capable of inducing such variations, which, even though initially small, may have cumulative effects in the economy in subsequent time periods.

DISCUSSION by JAMES N. MORGAN, University of Michigan—It seems to me that the theory of the motivation of people underlying many econometric models is inadequate. For instance, do people have some fixed level of savings or of durable goods (as Dr. Ulmer's model suggests) to which they aspire, or do their aspirations change with their achievement, as the psychologists insist?

Dr. Klein has reminded us that the parameters of a structural relation between x and y (e.g., consumption and income) cannot be found by simple regression if the parameters themselves affect the marginal distribution of the y's (the income distribution). He suggests that this difficulty can be handled by solving simultaneously another equation for the dependence of the marginal y-distribution upon the average y and therefore on x. But dependence can occur in the other direction, and if the parameters of the $x=f(y)$ relation depend on the marginal distribution of the y's (consumption function depends on the distribution of income), then the solution would be much more complicated.

The Haavelmo formula given by Klein handles differences between individuals' consumption functions by added variables relating to their characteristics, such as age, etc. But this will not do for such factors as liquid asset holdings where not only the level but also the *slope* of the income-consumption relation is influenced, i.e., where the factors are not additive.

[1] James S. Duesenberry, "Income-Consumption Relations and Their Implications," in *Income, Employment and Public Policy*, Essays in Honor of Alvin H. Hansen, New York: W. W. Norton and Co., 1948, pp. 54–81.

Remarks on the Aggregation Problem, F. W. Dresch, U. S. Naval Proving Ground, Dahlgren, Virginia.

A new aspect of the aggregation problem has arisen in connection with the input-output matrix technique in macro-economic theory. Here the most natural choice of index number appears to be the fixed-base type derived from simple aggregation of quantity variables measured in dollars-worth at base year valuation. The indices thus defined are weighted arithmetic means of relatives with base year value weights. As the interval of time (or displacement) spanned by the index approaches zero, these indices reduce to the Divisia continuous-chain index which uses shifting (current year) value weights. Further comparison of the properties of these fixed-base indices with the Divisia continuous-chain indices appears to indicate that, although the aggregation problem for a general micro-economic system involves simultaneous selection of a whole *system* of indices, the problem of choosing a *type* of aggregate for the system may not be essentially different from that of choosing a type for an isolated index. In both cases there may be no type "best" for all purposes. For descriptive purposes, such as in the input-output matrix approach, in which technical and market phenomena are somewhat intermixed, the fixed-base (dollars-worth) choice may be preferable. In case the technical situation is to be isolated and technical functions are linear in the logarithm of inputs and outputs (Cobb-Douglas type), Klein's indices (dependent on the technical functions) may be suitable. For explanation of the mechanisms determining economic variables, it may be more appropriate to employ Divisia indices defined as linear sums of price and quantity relatives (with current value weights) when these relatives are regarded as decision variables subject to probability distributions. These indices may be even more relevant if the micro-theory becomes a mixture of the classical general economic equilibrium and the institutional viewpoints, a merger including equilibrating mechanisms and factors of institutional growth, all operating through stochastic linkages.

DISCUSSION by James S. Duesenberry, Harvard University, and Avram Kisselgoff, University of Illinois. (No abstracts available.)

PROBLEMS IN THE THEORY OF INTERNATIONAL TRADE

James W. Angell of Columbia University presided over a session Wednesday afternoon, December 28, dealing with problems in the theory of international trade. Papers and prepared discussion were offered as follows:

Measurement of Price Elasticities in International Trade, GUY H. ORCUTT, Harvard University.

THE MAIN thesis of this paper is that statistically estimated price elasticities are unreliable for use in predicting the effects of a relative depreciation and probably considerably underestimate the effectiveness of such a depreciation. The following points are developed:

1. In most studies the relevant range of observed price variation is less than five or ten per cent of the average price level. This range is small relative to the magnitude of errors that are likely to be present in the data.

2. It is unreasonable to assume that changes in taste and in technology have been so minor over a twenty-year period as not to have shifted the demand surface enough to account for a substantial part of the small relevant price variation that took place. Shifts in the demand surface which were uncorrelated with shifts in the supply surface would result in an underestimation of price elasticities. Insofar as shifts of the demand and supply surfaces were not independent they were most likely positively correlated and this would also lead to underestimation of price elasticities. Any world-wide shifts in relative demands or technology will cause import demand and supply schedules to shift up and down together.

3. The magnitude of errors of observation must be large relative to the range of relevant price variation. This must lead to errors of estimation and very likely to substantial underestimation of price elasticities.

4. Historical price and quantity indices reflect price changes of commodities with very different price elasticities and it seems reasonable to assume that historical price changes have been largest for goods that have low price elasticities.

5. Short-run instead of long-run price effects have been estimated.

6. The price elasticity of demand for imports or exports is probably much larger for large price changes than for small price changes.

Some suggestions are made of ways by which some sources of error and bias might be removed, and it is pointed out that little light is likely to be thrown on the effectiveness of depreciation by grinding out more regression analyses on the same general type of time series.

Application of Statistically Derived Import Elasticities to Practical Problems of Foreign Trade Policy, RAYMOND F. MIKESELL, University of Virginia.

STATISTICALLY derived import demand elasticities have been applied to the following types of problems: (1) the estimation of the demand for imports for the United States and other countries on the basis of assumed

national incomes for the purpose of formulating national plans; (2) estimating the effects of changes in tariff rates on imports; (3) the determination of foreign exchange rates; and (4) problems relating to the dynamics of world equilibrium.

In general, it is concluded that income and price elasticities of demand for total imports are subject to serious error and are likely to be misleading. The derivation of an income demand function from the historical relationship between income and imports and of a price demand function from the historical relationship between imports and relative prices involves exceedingly hazardous assumptions. Moreover, the projection of historical relationships to periods following important structural changes in the world economy appears to be unwarranted. In addition, it is suggested that in present day markets characterized by imperfect competition imports tend to be influenced in considerable measure by the selling efforts and other export policies of the exporting countries.

It is suggested that studies be made of the behavior of individual commodity imports and categories of imports classified as to end use over periods of cyclical change. Similarly, studies of the probable reaction of import demand to changes in the prices of individual commodity imports by commodity specialists would be of considerable value for a number of problems. Finally, we need to know a great deal more about the probable reaction of foreign and domestic suppliers of individual commodities in order to reach conclusions on tariff and foreign exchange policies.

DISCUSSION by FRITZ MACHLUP, The Johns Hopkins University—Fear that the magnitudes of the relevant elasticities are such as to prevent the free-market mechanism from working satisfactorily may be called "elasticity pessimism." Elasticity pessimism in international-trade theory makes many economists advocate foreign-exchange restrictions. Most of them underestimate the actual price elasticities of demand for imports and exports, and overestimate the price elasticities required for exchange depreciation to have remedial effects. The *under*estimation of the *actual* elasticities is chiefly due to the statistical techniques and erroneous interpretations of data. Mikesell, Orcutt, and Harberger have enumerated several reasons why estimates have been too low. The *over*estimation of the *required* elasticities is due chiefly to two errors of reasoning: (1) It is wrong to regard unity as the "critical value" for the sum of the elasticities of demand for exports and imports, except if the elasticities of supply are infinite. With lower elasticities of supply, depreciation may have remedial effects even if the sum of the elasticities of demand is below unity. (2) Where initially an import surplus exists, the critical value of demand elasticities is still lower, because a percentage reduction in the value of imports will be absolutely more significant than a similar percentage reduction in the value of exports.

An International Economic System, T. C. CHANG, United Nations, and J. J. POLAK, International Monetary Fund. (Abstract published in "Compte-Rendu du Congrès de Colmar," ECONOMETRICA, Vol. 18, January, 1950, pp. 70–72.)

DISCUSSION by H. NEISSER, New School for Social Research—(a) The independent variable used in the actual estimate of equation (2) (see abstract in January issue) is not x_i [which, however, is to be retained in (1)], but x_i enlarged by such magnitudes as autonomous imports, etc. (b) The series used for measuring the principal exogenous variable (i.e., investment that is not a function of income), viz. building activity in the United States, is neither truly exogenous nor representative of exogenous investment. (c) The "equilibrium" values for x_i and m_i from the estimate would only by chance yield a trade balance that equals the available invisible items in the balance of payments, plus gold flow. Since in most countries these values are considered dependent only on exogenous factors outside of the country's control, especially on United States investment, his system will currently tend to collapse.

Flexible Exchange Rates and the Theory of Employment, SVEND LAURSEN, Williams College, and LLOYD METZLER, The University of Chicago. (No abstract available.)

DISCUSSION by GOTTFRIED HABERLER, Harvard University. (No abstract available.)

ANALYSIS OF THE MULTI-PART ECONOMY

Papers and discussion on the analysis of the multi-part economy were presented, as follows, in a late afternoon session on Wednesday, December 28. Harold M. Somers of the University of Buffalo served as chairman.

A Matrix Multiplier, R. M. GOODWIN, Harvard University.

LET A_{ij} denote payments to sector i from sector j, a_{ij} denote payments to sector i from sector j per dollar's worth of the current rate of payments, y_j denote the current rate of payments, and b_{ij} denote the payment to i from j not explained by y_j. Then

$$A_{ij} = a_{ij}y_j + b_{ij},$$
$$r_i(t) = \text{receipts of } i \text{ at time } t,$$
$$r_i(t) = y_i(t+1).$$

The basic matrix equation of payments is

$$\left\{\begin{matrix} y_1(t+1) \\ \\ \\ \\ y_n(t+1) \end{matrix}\right\} = \left\{\begin{matrix} \sum_j b_{1j}(t) \\ \cdot \\ \cdot \\ \cdot \\ \sum_j b_{nj}(t) \end{matrix}\right\} + \begin{bmatrix} 0 & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 0 & a_{n-1,n} \\ a_{n1} & \cdots & \cdots & a_{n,n-1} & 0 \end{bmatrix} \left\{\begin{matrix} y_1(t) \\ \\ \\ \\ y_n(t) \end{matrix}\right\}$$

For simplicity, we may write this as $y(t+1) = \beta(t) + ay(t)$, where β_i replaces $\Sigma_j b_{ij}$. The general solution (for a stable system) is

$$y(t+1) = \sum_{\tau=0}^{t} a^{\tau}\beta(t-\tau) = \sum_{\tau=0}^{t} a^{t-\tau}\beta(\tau).$$

The geometric matrix series resulting from a steady rate of spending of one dollar in all sectors is $I + a + a^2 + a^3 + \cdots$. This series converges to $[I - a]^{-1}$ if all the columns of a add to less than one, i.e., if the marginal propensity to spend in all sectors is less than one.

If a has n distinct roots (which we may assume), then there exists a matrix h such that $hah^{-1} = c$, where c is diagonal with the n roots along the diagonal. By this canonical reduction we may transform to generalized coordinates, η, each of which is independent of all others, so that

$$\eta_i(t+1) = \lambda_i\eta_i(t), \qquad (i = 1, 2, \cdots, n).$$

It is easy to prove that all these roots cannot be positive and real. Therefore the matrix multiplier *necessarily* has cyclical components. If we assume that all sectors have a marginal propensity to spend of less than one, all the roots lie within the unit circle in the complex plane, and hence all motions are stable. For constant injections, the system contains n separable, simple multipliers $1/(1 - \lambda_i)$, $(i = 1, 2, \cdots, n)$.

DISCUSSION by Everett E. Hagen, University of Illinois—The single most striking aspect of Professor Goodwin's paper is the demonstration that the matrix must oscillate. With constant repeated injections, this means (as Samuelson noted in conversation) oscillation about the path of approach to a new equilibrium, *not* oscillation about the new equilibrium. Oscillation in this limited sense occurs because as the pulses of spending pass through the matrix, in one time period they will stimulate induced spending in one group of cells; in another period, in another group; and the weighted average marginal propensity to spend during one period will thus (except by coincidence) differ from that during another. But this demonstration is less general than it seems. It depends in part upon the fact that an empirical matrix cannot have repeated latent roots. But a large empirical matrix may have roots *almost* equal, i.e., weighted average marginal propensities in different periods *almost* identical; hence the movement may be virtually nonoscillatory. This may be the common empirical case (in the absence of an acceleration effect, from which Goodwin implicitly abstracts). Yet Goodwin's proposition is useful for studying various cases where this is not true. (I am obligated to Professor Hurwicz for an explanation of Goodwin's mathematics; but he is of course in no sense responsible for any error in my comments.)

The Multi-Part Multiplier, John S. Chipman, The Johns Hopkins University. (Submitted for publication in a subsequent issue of Econometrica.)

DISCUSSION by William C. Hood, Cowles Commission for Research in Economics, The University of Chicago—The papers by Chipman and Goodwin gen-

eralize earlier results in the field obtained in particular by Machlup, Metzler, and Samuelson. Goodwin's theorem concerning the necessity of oscillations in the income movement associated with the matrix multiplier appears to be stronger than many economists would approve for the general case, based as it is on the assumption that each sector's marginal propensity to spend internally is zero. Neither paper recognizes explicitly that stability of income movements may be consistent with a total marginal propensity to spend greater than one in some sectors. This result was given by Metzler for the two-economy case in 1942. Both papers contribute valuable theorems on stability, monotonicity, or oscillation of income movements and demonstrate clearly the formal analogy with the simpler, more aggregative multipliers usually studied.

Measurement of Regional Income Flows, Werner Hochwald, Washington University.

An analysis of the multi-part economy by regional sectors presupposes the measurement of regional income flows. This measurement is difficult due to a serious lack of regional income data, regional expenditure data, and data on the interregional flow of goods and money. While many corresponding data are available for the national economy as a whole, compiled largely from administrative records, these data are missing for individual regions because the basic administrative records, such as export and import statistics, are not kept by regions.

The lack of regional data can be overcome in four ways: First, national data may be allocated to regions in accordance with certain related information, available either for some components of the aggregate data, such as covered payrolls, or available for some benchmark periods, such as Census data; this, for example, is the method used for deriving the official state income payment estimates compiled by the U. S. Department of Commerce. Second, regional data may be derived from sample surveys designed specifically for the purpose of estimating a regional rather than national population; this, for example, is the method used for compiling the department-store sales index compiled by the Board of Governors of the Federal Reserve System. Third, a balance of interregional trade may be based on transportation statistics and appropriate pricing of commodity movements between regions; this, for example, is the method used in P. C. Hartland's recent study on the balance of interregional payments of New England. Fourth, a balance of interregional payments may be based on a study of money flows between regions, such as shown in the net credit or debit of each Federal Reserve Bank in the Interdistrict Settlement Fund.

The net balance of interregional payments must be the same under any of the four methods employed. Thus, the net balance can be used to test the internal consistency of independent regional estimates, arranged in a system of regional income and expenditure accounts. Such a system has

been developed for the Eighth Federal Reserve District where, for 1948, the personal income and expenditure accounts show a net outflow of funds amounting to 2.5% of total income payments, as contrasted with a net balance of 0.5% for a corresponding system of national income and expenditure accounts.

DISCUSSION by ROBERT V. ROSA, Federal Reserve Bank of New York, and RUTLEDGE VINING, University of Virginia. (No abstracts available.)

INTERPRETATION OF INDEX NUMBERS

An early morning session on Thursday, December 29, was devoted to the presentation of the following two papers on the interpretation of index numbers. William Vickrey of Columbia University served as chairman.

The Accuracy of Index Numbers, BRUCE D. MUDGETT, University of Minnesota.

INDEX numbers which compare price or quantity changes between two periods (t_0 and t_1) can be classified in a double trichotomy on the basis of the following.

A (the source of information used): (1) an index may use both price and quantity information from both periods (p_0, q_0, p_1, q_1), or (2) only part of this information (e.g., p_0, p_1, q_0 but not q_1), or, finally, (3) it may use information other than that found in periods t_0 and t_1; and

B (extent to which information from a source is used): (1) they *should* be constructed from all of the goods sold in both t_0 and t_1, (2) by known methods they must be constructed from the data of prices and quantities of those goods that are present in both periods (excluding information about goods present in one period and not in the other), and (3) they are generally based upon a sample from (B.2) above.

Indexes of the form (A.1) are best, those of form (A.2) are logically acceptable but are only approximations, those of form (A.3) are logically unacceptable.

There are three kinds of error in any index number. There is a sampling error due to the use of data of (B.3) in place of (B.2). There is a formula error due to differences among the best indexes, those of form (A.1). Finally, there is a homogeneity error due to the use of data of (B.2) in place of (B.1). There is still an additional error if the index number is of the form (A.3). The conclusion is that the most acceptable formulas must be of form (A.1.). There may, of course, be error also in the original price and quantity data.

The recommendation is then to construct link indexes of form (A.1) between consecutive periods. This gives the most exact measurement among known methods. For a series comparison, the links are then to be chained together as first proposed by Marshall in 1887. This procedure minimizes the homogeneity and formula errors which are serious sources of inaccuracy in direct comparisons between distant periods.

Index Number Problems in International Trade, FRANCIS E. McINTYRE, California Texas Oil Company, Ltd.

IT IS of course elementary that in times of currency inconvertibility conventional demand studies produce almost totally irrelevant results with respect to imported commodities. Of much greater importance are the balance of payments position of the intended market and the ability of the would-be marketer to use the currency which is thus determined to be available. Under these circumstances, the importance of assessing as promptly as possible the prospective currency earnings of any country will be obvious. Unfortunately for such efforts at assessment, the necessary data are somewhat elusive.

In the light of recent experiences, I am satisfied that the greatest need of our profession is an amalgamated order of statisticians which can unite the citizens of the world to prevent a recurrence of devaluation. It is difficult to believe that fiscal benefits can possibly ensue which will offset the statistical chaos which confronts us. Devaluation has served only to highlight a number of difficult problems in the use of index numbers across State boundaries. Wherever exchange rates are a factor in the index number calculation, the resulting indexes are subject not only to the whims of devaluation, but also to many other conversion problems. The import and export statistics on which the index numbers of concern to us are based are likewise harassed by the "consumption entry" question. The effect upon the payments balance of different kinds of trade makes the inclusion or exclusion of some data a moot issue. Changes in the methods of collection of taxes may well determine whether these taxes are included in export and import prices used for index number calculation.

CONTRIBUTED PAPERS

A joint session of the American Statistical Association, Institute for Mathematical Statistics, and Econometric Society was held Thursday morning, December 29, for papers of particular interest to statisticians. Harold T. Davis of Northwestern University presided as chairman, and the following papers were given:

Simple Regression Analysis with Autocorrelated Disturbances, HOWARD L. JONES, Illinois Bell Telephone Company.

WHEN the disturbances in a regression equation are connected by a linear difference equation, the parameters of both equations can be estimated simultaneously by maximizing a function that describes the joint probability of the disturbances or linear functions thereof. Two cases are of particular interest: (1) the case where the distribution of the disturbance associated with the earliest observation is unknown; (2) the case where the distribution of this disturbance is implied by the specification that the linear relationship between the disturbances holds not only for the period observed, but for all prior periods as well. The form of the estimates and some practical applications suggest that the possibility of autocorrelation may well be considered in problems where a regression equation appears to be improved by including lagged variables.

Application of Sequential Sampling Method to Check the Accuracy of a Perpetual Inventory Record, JOSEPH B. JEMING, New York, New York. (No abstract available.)

A Test of Klein's Model III for Changes of Structure, A. W. MARSHALL, The RAND Corporation.

A GOODNESS of fit test for single structural equations from linear stochastic equation systems is derived. This result is based upon H. Rubin's expression for the approximate distribution of the calculated residuals of these equations:

$$\mathcal{E}(u_t^*) = 0,$$
$$\mathcal{E}(u_t^{*2}) = \sigma^2 + \sigma^2/N + (\sigma^2/N)\, z_t^* M_{z^*z^*}^{-1} z_t^{*\prime} + \operatorname{tr} \Delta\Omega + z_t^0 \pi^{**\prime} \Delta \pi^{**} z_t^{0\prime}.$$

These quantities or estimates of them are available and test regions having appropriate characteristics can be constructed under certain normality assumptions. Use is made of the properties of naive economic models ($y_t = y_{t-1} + v$ and $y_t = y_{t-1} + \Delta y_{t-1} + v$) to truncate these regions by a *reductio ad absurdum* argument in such a way as to limit the absolute size of residuals which will be accepted. This truncation is effective only for observations rather far removed from the mean of the observations of the computation period.

This test was performed on the Klein Model III for the years 1946 and 1947. The results essentially were that equations C, M_2, ΔY fit rather badly, equations H_1, D_2, v, Δr, Δi fit very well, and the remaining equations rank roughly in order of goodness of fit D_1, M_1, I, W_1.

Application of the Theory of Extreme Values to Economic Problems, S. B. LITTAUER, Columbia University, and E. J. GUMBEL, New York, New York.

INSTEAD of the usual search for regularity in economic time series, patterns of irregularity are sought. The apparent cyclical recurrence is better interpreted as a sequence of statistically different universes which together might make a super-universe providing possible prediction with respect to aggregates of events, but little useful inference with respect to particular events. A methodology needed to interpret these phenomena is contained in statistical quality control and in the theory of extreme values.

Very simple applications of control charts show little homeostasis in economic time series, at best stability for only short time intervals, whereas it seems definite that these phenomena are stochastic in the large. The maxima of the Dow-Jones index, interpreted in the same way as flood frequencies, show that the 1928 peak value, being completely outside of the previous levels, ought to have led to a forecast of the imminent breakdown. The yearly ranges of the Ayres index per month for 1839 to 1937 clearly follow the asymptotic distribution of the range.

Bias Due to the Omission of Independent Variables in Ordinary Multiple Regression Analysis, T. A. BANCROFT, Iowa State College.

GIVEN n observations of the dependent variate y and the independent variates $x_1, x_2, \cdots, x_k, \cdots, x_r$, $k < r$, all variates measured from their respective sample means, we have calculated the ordinary regression of y on the first k variates and y on all r variates. We define ordinary multiple regression as the single-equation approach, error only in y which is assumed normally and independently distributed with zero mean and variance σ^2, the x_i being fixed from sample to sample.

In order to determine whether to omit or retain the last $(r - k)$ independent variates we formulate a rule of procedure: calculate Snedecor's $F = \dfrac{\text{reduction in } Sy^2 \text{ due to } (r-k) \text{ variates } /(r-k)}{\text{error mean square after fitting all } r \text{ variates}}$. If F is nonsignificant at some assigned significance level α we pool the sums of squares and degrees of freedom involved in the numerator and denominator of F to obtain an estimate of the error σ^2, and fit y on the first k variates only. If F is significant at the assigned significance level we use the denominator only in F for our estimate of σ^2 and hence fit y on all r variates.

The object of this investigation is to determine the bias in our estimate e^* of σ^2 if we follow such a rule of procedure. The bias turns out to be

$$\frac{2\sigma^2\lambda}{n_1 + n_2} + \sigma^2 e^{-\lambda} \sum_{i=0}^{\infty} \left[I_{x_0}\left(\frac{n_2}{2} + 1, \frac{n_1}{2} + i\right) - I_{x_0}\left(\frac{n_2}{2}, \frac{n_1}{2} + i\right) \frac{-2i}{n_1 + n_2} I_{x_0}\left(\frac{n_2}{2}, \frac{n_1}{2} + i\right)\right] \frac{\lambda^i}{i!},$$

where $x_0 = n_2/(n_2 + n_1\alpha)$, $\lambda = \Sigma_{i=k+1}^{n} (\beta_i')^2/2\sigma^2$, n_1 and n_2 are the respective degrees of freedom for the numerator and denominator of F, and $\Sigma_{i=k+1}^{r} (\beta_i')^2$ is a function of the population regression coefficients $\beta_{k+1}, \cdots, \beta_r$.

Estimating Parameters of Pearson Type III Populations from Truncated Samples, A. C. Cohen, Jr., University of Georgia.

The method of moments is employed with "single" truncated random samples (1) to estimate the mean, μ, and the standard deviation, σ, of a Pearson Type III population when α_3 is known and (2) to estimate μ, σ, and α_3 when only the form of the distribution is known in advance. No information is assumed to be available about the number of variates in the omitted portion of the sample. The results obtained can be readily applied to practical problems with the aid of "Salvosa's Tables of Pearson's Type III Function." An illustrative example is included in the paper.

The Cyclical Normal Distribution, E. J. Gumbel, New York, New York.[1]

The usual normal distribution becomes invalid for variates, like an angle, lying on the circumference of a circle. The distribution of such variates was established by R. von Mises by the same methods as used for the classical derivation. The cyclical normal distribution is symmetrical about a mode and antimode. The probability function is proportional to an incomplete Bessel function of the first kind and of order zero for an imaginary argument, and contains two parameters, the direction of the resultant vector and a parameter k linked to the absolute amount of the vector. The parameters may be estimated by the method of maximum likelihood. For $k = 0$, the distribution degenerates into a uniform cyclical distribution. If k is of the order 3, the distribution approaches the linear normal one, k being the reciprocal of the variance. With increasing values of k, the distribution loses its cyclical character and becomes concentrated in a narrow strip. This distribution holds for symmetrical values varying according to pure chance about a unique

[1] Work done in part under contract with the Research and Development Branch, Office of the Quartermaster General.

mode in a closed space (as the angles of the wind directions) or a closed time, and gives a theoretical model for the variations of temperatures, pressures, rainfalls, storms, discharges, floods, death and birth rates over the year, and earth quakes over the day. The comparison between theory and observations in plotting the square roots of the frequency on polar coordinate paper provides a statistical criterion for the regularity of cyclical phenomena.

LINEAR MODELS OF PRODUCTION AND ALLOCATION

A session on linear models of production and allocation was held jointly by the American Economic Association, American Statistical Association, and Econometric Society on Thursday afternoon, December 29. The following papers and discussion were presented under the chairmanship of Solomon Fabricant of New York University.

Leontief's System in the Light of Recent Developments, NICHOLAS GEORGESCU-ROEGEN, Vanderbilt University. (No abstract available.)

Efficient Allocation of Resources, TJALLING C. KOOPMANS, Cowles Commission for Research in Economics and The University of Chicago. (Submitted for publication in a subsequent issue.)

DISCUSSION by MARSHALL K. WOOD, United States Air Force—Since 1946 the Air Force has been developing linear models as tools for program planning within the Air Force. Data have been collected and test problems computed; practical use for operating purposes is imminent. Our computing techniques give solutions to dynamic systems of fifty activities and thirty-six time periods in about one day. The time required for larger systems is approximately proportional to the number of activities and time periods. The model used is dynamic, analogous to the von Neumann model although independently developed, hence similar to one discussed by Koopmans. It is similar to the Leontief dynamic model, the principal differences being replacement of differential equations by finite difference equations, introduction of time lags and storage activities, and use of a triangular structure to facilitate computation. The Air Force has also been interested in the development of dynamic models of the civilian economy for testing the feasibility of military programs, and is supporting Professor Leontief's work. Nonlinear relationships may be expected in the capital equipment and manpower segments of this model, and will be incorporated whenever empirical evidence makes it necessary. Computing techniques have been studied which can accommodate such nonlinear relationships without excessive difficulty.

DISCUSSION by WASSILY W. LEONTIEF, Harvard University, and GEORGE J. STIGLER, Columbia University. (No abstracts available.)

CONTRIBUTED PAPERS

The following papers were presented in sessions late Thursday afternoon, December 29, under the chairmanship of Milton Friedman of The University of Chicago, and early Friday morning, December 30, under the chairmanship of Sidney S. Alexander of the International Monetary Fund.

Differential Effects of Taxation in the Trucking Industry, RICHARD W. LINDHOLM and VICTOR E. SMITH, Michigan State College.[1]

IN THIS study of the equality of the burden of fuel and oil taxes paid by specialized truckers, four types of truckers were considered: household goods, heavy machinery, liquid petroleum products, and retail store delivery carriers. All Class I firms in 1947 which were also in Class I in 1945 were included.

We have defined a tax differential as any tax payment that departs from the average payment by truckers of all four types, if receiving the same operating revenue. To reduce state-to-state variations in fuel tax gallonage rates, the firms were classified into regional groups. In each region we estimated the average tax at each level of operating revenue by a least-squares regression of fuel and oil tax payments on operating revenues. The deviations from this regression line are our measures of the tax differentials of the various firms.

The differences among the mean tax differentials of the various types of truckers were highly significant in every region except the Southwest. (There only nine firms of the total of fifty-eight were not heavy machinery haulers.) The magnitude of the differentials is indicated by the figures for the Central Midwest region, where the range was the greatest. The mean for liquid petroleum carriers there was +$9353 while that for household goods carriers was −$8824. In each region these two types of carriers were, respectively, most heavily and least heavily taxed. The smallest mean differential paid by liquid petroleum products carriers was +$4244.

If such tax differentials occur often where uniform specific taxes on input are levied, investigations into their possible effects on operations and the allocation of resources are indicated. For this purpose, of course, knowledge of the cost and revenue functions is needed.

[1] Supported in part by the Ohio State University Research Foundation. Acknowledgement is due Mr. William Jr. Thomas for carrying through the calculations.

The Application of Linear Programming Methods and the Simplex Technique to the Hitchcock-Koopmans Transportation Problem, George B. Dantzig, United States Air Force.

The purpose of the paper was to discuss the following: Each of m locations has given amounts of a homogeneous product to be shipped out. Each of n other locations has given amounts of the same product to be shipped in. It is possible to ship from any location having the product available to any location requiring the product. The problem is to choose such routes and amounts to be shipped along these routes to minimize the total cost of transportation where the cost to ship is proportional to the shipped amount and the constant of proportionality depends on the route and is known for all possible routes.

Production Scheduling for Monopolized Products, Robert Dorfman, Operations Analysis, United States Air Force.

Linear programming permits a more realistic analysis of the behavior of many types of firm than previous methods. It applies particularly to industrial firms which have only a finite number of processes for manufacturing their outputs, and for which output is a linear function of variable costs. But previous expositions of linear programming have assumed that the objective function of the firm was linear; in more usual language, that net profit was a linear function of total variable costs. This assumption is satisfied, at best, only by firms which are in perfect competition with regard to both sales and purchases.

The methods of linear programming can be applied quite readily to two more complicated marketing situations, namely, to monopolists who face linear demand curves and to monopsonists who face linear supply curves. Since both of these cases have the same theoretical structure it is sufficient to discuss the monopolistic one.

Assume that the physical output of such a monopolist is a linear function of total variable expenditures on the finite number of industrial processes available to him. Then it can be shown that net profit is the value of a quadratic function whose independent variables are the expenditures on the several processes. Straightforward calculation of partial derivatives with respect to these variables will not, in general, lead to a practicable optimum production program, but a variation of the methods of linear programming will.

Thus it is seen that linear programming is a powerful new tool for the analysis of monopoly and monopsony and, indeed, of all situations where the utility of each unit of output (or input) is a linear function of the level of output. Still further generalization may be possible. The method is particularly applicable to multi-product situations.

The Producer's Cost Function in Comparative Statics: Application to Some Basic Industries, MICHAEL J. VERHULST, International Bank for Reconstruction and Development.

GRAPHS relating to four industries: airlines, railways, manufactured gas, and petroleum, show that, in the first approximation, the cost of production y of each company in a given year, the corresponding output x, and the capital h of the company satisfy a linear and homogeneous function:

$$y = K_1x + K_2h, \tag{1}$$

where K_1 and K_2 are constants for each industry. The main purpose of the paper is to give a theoretical explanation of this empirical relationship.

It is shown, first of all, that, for a given enterprise, the output is a function of the degrees of capacity utilization of the different organs of the enterprise. Assuming, then, that the entrepreneur knows in advance the price that he must pay for the factors of production, given the quantities of factors that he wants to buy, it is shown that, if y denotes the minimum cost of production of the output x, we have

$$y/y_0 = \varphi(x/x_0) \tag{2}$$

where φ is a given function and y_0 and x_0 are constants, y_0 being the minimum cost of production of the output x_0 corresponding to the normal capacity of the enterprise. But as, in the first approximation, both y_0 and x_0 are good indices of the capital h of the enterprise, we see that

$$y/h = \varphi'(x/h), \tag{3}$$

where φ is the firm's cost function, h being a constant for the enterprise. It just happens that we know empirically the form of the function (3). It is:

$$y/h = K_1(x/h) + K_2, \tag{4}$$

which is the same as (1).

This function shows the way in which the equilibrium positions (y, x) will change as a result of changes in the parameter h taken as independent parameter. It is therefore a producer's cost function in comparative statics. Many Cobb-Douglas production functions which have been computed are actually of a similar type but in which the cost of labor is taken as an index of the cost of production.

Welfare Losses, ROGER DEHEM, University of Montreal.

AN INDEX of inefficiency is required to measure the waste caused by any economic policy or institution. But there exists no unique set of

criteria to define an optimum welfare state. Optimal production and exchange conditions, as defined by, among others, V. Pareto, E. Barone, J. R. Hicks, and M. Allais, leave the allocation solution partially indeterminate. The deviation of any actual allocation from any of these optimal points, whose coordinates are the elements of an allocation matrix in the space of individuals and available goods, can be measured in terms of any single good or combination of goods. The set of all alternative measures of waste is obtained by moving from the actual intersection locus of individual indifference hyper-surfaces to all possible loci of tangency of these surfaces (assumed to be convex).

Assuming the existence of a welfare index, as defined by A. Bergson, O. Lange, and P. A. Samuelson, an index of distribution adequacy or inadequacy can be defined for every level of the index of Paretian inefficiency. The welfare loss due specifically to maldistribution can thus be measured in the same way as the loss due to inefficiency in the Paretian sense, by computing the difference between the quantities of goods actually used to achieve a given welfare level and the set of all quantities that would achieve the same welfare state if optimally distributed, the Paretian efficiency of the allocation remaining unchanged.

Ancient Prices and Their Interpretation, HAROLD T. DAVIS, Northwestern University. (No abstract available.)

The Universal Discount as a Means of Economic Stabilization,[1] J. G. BAKER, Baker Manufacturing Company, Evansville, Wisconsin.

UNDER present conditions, because of the marked correlation of employment and price level during a sustained rate of change in either, anticipated price changes at such a time influence decisions conducive to economic instability. The appearance of instability from time to time in the form of cumulative upward and downward movements in employment and price level is therefore not surprising.

The universal discount plan appears to provide the means of changing the timing and amount of the movements of the price level in relation to the movements of employment in such a way that anticipated price changes would influence decisions conducive to stability. It is therefore expected that the universal discount plan would replace the present cumulative effect with a damping effect capable of smoothing out business cycle variations.

Contrary to what one might expect, the universal discount plan does not involve price or wage dictation. Markets would be free. The universal discount plan also avoids the use of investment control, currency

[1] A revision and extension of a paper published in ECONOMETRICA, Vol. 16, April, 1948, pp. 155–184.

management, pump priming, tax manipulation, and an active monetary policy.

The universal discount plan by design depends on a simple statutory formula rather than arbitrary administrative action. This avoids both the ideological objections and the political and administrative difficulties of commonly advocated stabilization methods.

The Structure of Interest Rates, William S. Morris, Hirsch and Company, New York, New York.

The following model of the bond market was presented: (1) For *taxable securities* the yield equals $2^{r/m} \cdot 2\frac{1}{2}\% \cdot \tanh(a + bn)$, where r denotes the credit variable (a nonnegative integer), m the market variable (also a nonnegative integer), n the years to maturity, and a and b are selected so that $2\frac{1}{2}\% \cdot \tanh(a + b)$ equals the yield on one-year treasury certificates and $\tanh(a + 2b)/\tanh(a + b) = 2^{1/m}$. (2) For *tax exempt securities* the yield equals $2^{(r - t/2)/m} \cdot 2\frac{1}{2}\% \cdot \tanh(a + bn)$, where r and m are used in the same way as for taxable securities, t denotes value of tax-exemption for long term securities (a nonnegative integer), and a and b are selected so that $2\frac{1}{2}\% \cdot \tanh(a + b)$ equals about 55% of one-year government yield and $\tanh(a + 2b)/\tanh(a + b) = 2^{1/m}$.

Production with Limitational Goods, John S. Henderson, University of Alabama.

According to the theory of marginal productivity, a competitive firm will extend its use of a variable productive service until its price equals the value of its marginal product. Mathematically, this is expressed by the equation

$$p_s = (\partial Q/\partial x_s)\, p_q, \qquad (s = 1, 2, \cdots, r), \tag{1}$$

where Q is the quantity of product, x_s is the quantity of the service required, p_s is the price of x_s, and p_q is the price of the product.

Pareto, Georgescu-Roegen, and others have pointed out that the marginal product, $\partial Q/\partial x_s$, might not exist because its existence is postulated on the supposed existence of a single production function:

$$Q = Q(x_1, x_2, \cdots, x_r)\,. \tag{2}$$

Technical conditions of production preclude the existence, in general, of such a simple description of productive processes. An alternative procedure will yield more realistic results. Suppose the quantity of the sth good or service is x_s and its price is p_s, s running from 1 to n. Then we can write profit or $V = \Sigma_{s=1}^{n} p_s x_s$, where products x_s have a positive sign while productive services have a negative. Rather than assuming the existence of production function (2), it is appropriate to assume the

existence of m simultaneous technical equations each of which presupposes a technical relation between the x_s:

$$(3) \qquad {}_iq(x_1, x_2, \cdots, x_n) \qquad (i = 1, 2, \cdots, m), (m < n).$$

If V is maximized subject to (3), the result is a certain set of first and second order conditions, a set which probably gives a more accurate picture of profit maximization than does the marginal productivity theory. As a sample, the first order conditions may be written in the form:

$$(4) \qquad p_j + \sum_{i=1}^{m} p_i \frac{\partial x_i}{\partial x_j} = 0, \qquad (j = m + 1, m + 2, \cdots, n).$$

An Index Measuring the Degree of Reciprocity in International Trade Relations, Louise Sommer, Aurora College.

Assume that a group of countries (exemplified by the ERP countries) agree to lower the tariff rates against each other. How shall the degree of concessions given and concessions received be determined? Assume that, for example, Norway has decided to lower her tariff rates regarding her trade partners, one of which is Belgium. The decisive criterion would be the *intensity of interchange* between the two countries. This reciprocal trade intensity is measured by indices. The index number is calculated by the following formula: The percentage Norway's export to Belgium bears to Norway's total export is multiplied by the percentage Norway's export to Belgium bears to Norway's import from Belgium. A similar formula is applied to Norway's other trade partners. The resulting index numbers are ranked and become thus indicative of the reciprocal intertwining of the trade relations between Norway and her customer countries. The highest index numbers characterizing the best customer give a claim to an important discount. Concessions granted by Country A to Country B are thus made dependent upon the readiness of Country B to increase her purchases in Country A. Thus, an automatic link would be established between a gradual reduction of tariff rates and the reciprocal integration of trade within a group of countries (intra-European trade).

BUSINESS CYCLE PATTERNS

The following papers and prepared discussion were offered at a Friday morning session, December 30, at which Garfield V. Cox of The University of Chicago served as chairman.

Recent Patterns of Employment and Unemployment, Nathan Morrison, New York State Division of Placement and Unemployment Insurance.

The adoption of the federal old-age insurance law in 1935, and the passage of state unemployment insurance laws about the same time, created a new and rich source of data on employment and unemployment in this country.[1]

Since 1937, when the programs became effective, various regular patterns and relationships have been revealed in the employment and unemployment data. Among these are: (1) In New York State, during the ten years from 1939 to 1948, the average number of weeks of work per person each year was about 35. The range was narrow, from 34.2 weeks in 1942 to 36.4 weeks in 1939. (2) In 1937, there were 33,000,000 persons in this country employed in firms covered by the old-age insurance law; 40% of these persons were employed in covered employment in all of the ten years from 1937 to 1946. (3) Of all the persons with some employment during a year, 61% had employment in all four calendar quarters. (This was the average ratio for the eleven-year period 1938–1948 with a small range of variation.) (4) About 70% of the workers in New York State have one employer in a year, 15% have two, and 15% have three or more. (5) Each year in New York State an invariant relation has been found in the duration of unemployment: about 94% of the persons who have been unemployed for X weeks remain unemployed for at least $X+1$ weeks. (6) Of the group of persons who experience unemployment in a given year, about half have some unemployment in the following year; about a quarter have unemployment in a third successive year; and an eighth of the group have some unemployment in four years in a row. (7) About 55% of the persons who experience unemployment in a year have only one spell of unemployment; 25% have two spells; and 20% have three or more spells.

Continued intensive analysis of such types of patterns and relationships, using discriminant functions and the method of canonical correlations, may lead to the development of a comprehensive theory to coordinate and explain the observed regularities.

Methods of Measuring Absolute and Relative Variations in the Duration of Business Cycles, Nilan Norris, Hunter College.

Data pertaining to the duration of business cycles more closely approximate a series of historical events, rather than a sample of independent

[1] A detailed description of the material arising out of the operations of these social insurance programs has been given in an article by Nathan Morrison on "By-Product Data and Forecasting in Unemployment Insurance," *Journal of the American Statistical Association*, Vol. 44, September, 1949, pp. 397–405.

items drawn from a population in the sense implied by the modern theory of estimation. Although probability statements should be avoided unless conditions necessary for sampling are fulfilled, there may be advantages of using measures of absolute and relative variation which have a reasonably high degree of efficiency as provided by certain nonparametric methods usually involving order statistics. Of the various nonparametric measures, the range and the coefficient of rank correlation appear to be especially useful for analyzing variations in the duration of cycles. Less promising, but possible alternatives, are some of the parametric methods which provide asymptotically efficient estimates by means of the method of maximum likelihood. If the commonly used logarithmic normal curve is properly fitted to the frequency polygon of duration of cycles, the maximum-likelihood estimates of absolute and relative duration, respectively, are the standard deviation of the logarithms of the variates and the coefficient of variation of the logarithms of the variates. If the frequencies of duration of cycles are well graduated by a Pearson Type III curve, the maximum-likelihood estimates of dispersion are the arithmetic-geometric range, or $A - G$, for absolute variation, and the arithmetic-geometric ratio, or A/G, for relative variation, where A is the sample arithmetic mean and G is the sample geometric mean. An advantage of nonparametric methods over parametric methods is that the former involve no assumption about the form of the parent universe except that the cumulative distribution function is continuous.

Most of the work done by the National Bureau of Economic Research on variability of cycle lengths has been from the point of view of simple description, rather than estimation in the sense of sampling theory. The Bureau has used the average deviation, range, standard deviation, and variance to measure absolute variations in the lengths of cycles, and the coefficient of variation to measure relative variations. The Bureau has also used the coefficient of rank correlation to compare variations in the duration of cycles over different time periods.

Distribution of Turning Points of Time Series, C. ASHLEY WRIGHT, Standard Oil Company. (No abstract available.)

Cyclical Price Flexibility in Steel, CARL KAYSEN, Harvard University.

THIS paper presents for examination a simple economic model, designed to exhibit some of the problems arising from reliance on marginal cost pricing over the cycle as a regulator of capacity in an industry producing durable investment goods under cost conditions similar to those of the steel industry. The basic elements of the model are:

1. Abstracting from the influence of the price of steel, the economy is

subject to cyclical fluctuations in income of a steady sort, describable by a suitable Hansen-Samuelson difference equation.

2. Steel is supplied by a small number of firms, each having a short run cost curve characterized by constant marginal cost up to some fixed output, and sharply rising marginal costs thereafter. The level of the flat part of the curve, and the rate of rise of the steep part, differs from firm to firm; i.e., all firms are not equally efficient.

3. The demand for investment in any period is influenced by the price of steel in the previous period, if that price is high enough.

4. Firms which make an aggregate loss over the cycle withdraw from the industry, removing their capacity. If the marginal firm makes a high enough profit over the cycle, there is entry with addition to capacity.

These assumptions lead to a nonlinear difference equation of the second order, which is not soluble in general terms. Computations of specific solutions, with assumed "realistic" values of the parameters, show two kinds of results. If entry responses to profits are prompt, capacity is reduced, but not so much as to reduce the average level of income over the cycle, though enough to choke down its amplitude of fluctuation. If entry responses are not prompt, however, the average level of income as well as the amplitude of fluctuation are both reduced. These are equilibrium modes of behavior, reached only after several cycles are completed.

DISCUSSION by THOR HULTGREN, National Bureau of Economic Research—Mr. Morrison's paper is welcome evidence that social security data begin to be exploited. At first glance his figures on the number of weeks worked per worker startle one. They are low even in good years. But noncovered employment is absent from the numerator, and persons in the labor force only part of the year are counted full-time in the denominator. The lack of cyclical decline is more mysterious. Apparently the number of persons who receive no covered employment in a poor year and so drop out of the denominator is large.

Mr. Norris doubts whether known durations of cycles should be regarded as samples of a universe. I sympathize, but question whether refining the measurement of scatter would be profitable. Better to go on directly to ask: Why are some cycles longer than others?

Mr. Wright's approach to forecasting is original and ingenious. One of his "failures" is associated with an explicable peculiar distribution of turns during a war. In wartime some activities are discouraged, others stimulated. The former attain peaks early, the others late. The accompanying and following inflation creates a scarcity of peaks in price and dollar volume series. Wright suggests interesting further questions. I would like to add a few, less technical ones. Why does the transition from expansion to contraction, and vice versa, spread so gradually from industry to industry? In what practical business decisions are such forecasts useful? What would happen to cycles and the economy if their use became general?

DISCUSSION by ZENON SZATROWSKI, University of Buffalo—Mr. Wright proposes a simple objective method for forecasting the turning points of cycles. He

considers applications to the business cycle, but his technique appears particularly suitable in estimating turning points of common stock prices, where better data are available. The proposed method makes use of only a small proportion of the information usually available, but an extension of the idea could of course remedy this shortcoming. Following Wright's technique, one would observe the development of time series, note their turning points, and use a count of the early turns to estimate the month when most of the time series turned. Why not use other information which can be obtained in observing the behavior of time series? For example, in considering the peak of the cycle, the amount and "reliability" of the declines, as well as the number of turning points, can be measured and might be used as in weighting the turning points. In addition to the various observed behavior characteristics, we may also know about relationships among the individual time series. Incorporating this information would improve results, whereas ignoring it would actually raise questions about the applicability of the particular formulas used in estimating the parameters of the distribution.

DISCUSSION by Elmer C. Bratt, Lehigh University—The analysis of employment data can be made most useful if logical relations are kept in mind. For instance, I should like to know what offsetting forces are present to produce stability in the weeks of work per person.

The distribution of turning points in time series may have forecasting significance, but I am not convinced that Mr. Wright's frequency distribution method best reflects it. The needed interpretation would vary if substantial differences in the skewness of the curve of turning-point dates should occur from cycle to cycle. Because of the importance of erratic movements, I cannot agree that turning points in individual series can be satisfactorily identified shortly after they occur. Mitchell and Burns were similarly critical. Mr. Wright's study appears to indicate that the distribution of turning points does not differ substantially when leading and lagging series are separated. If this is true, regularity of pattern is not indicated. Other behavior patterns, such as amplitude and age classifications, do exhibit regularity. Perhaps emphasis should be shifted away from timing behavior patterns.

DISCUSSION by Clark Warburton, Federal Deposit Insurance Corporation—I believe we should abandon the concept of the business cycle as a continuous rhythmic process, and should regard cases of substantial unemployment or price inflation as recurring episodes of a pathological character. In studying such a condition the National Bureau procedure of examining a host of collateral phenomena is a common initial stage. In most investigations this procedure leads to concentration on a few key variables, and finally to detailed analysis of the character of the causal sequence. The rest of the phenomena then fall into three groups: predisposing conditions, symptomatic conditions, and irrelevant conditions. Sometimes investigators are baffled, usually when the key variable has escaped observation. This seems to be the status of studies of the depression-inflation malady. In this situation, further analyses of the symptoms, such as those by Mr. Norris and Mr. Wright, may improve the promptness and accuracy of diagnosis when another case of the malady occurs. However, their conclusions—such as that of Wright that we can be confident of the diagnosis three times out of four when a substantial percentage of the symptomatic conditions have been observed—are not very illuminating.

DISCUSSION by Robert W. Burgess, Western Electric Company, New York, New York. (No abstract available.)

ECONOMICS AND TECHNOLOGY: PRODUCTION FUNCTIONS

The concluding session of the Econometric Society was held Friday afternoon, December 30, jointly with Section K of the American Association for the Advancement of Science, and dealt with the general topic of production functions. Tjalling C. Koopmans of the Cowles Commission for Research in Economics served as chairman in the absence of Howard R. Bowen of the University of Illinois. Papers and prepared discussion were offered as follows:

Engineering Production Functions, WALTER RAUTENSTRAUCH, Columbia University. (No abstract available.)

Textile Production Functions, Equipment Requirements, and Technological Change, ANNE P. GROSSE, Harvard Economic Research Project.

THIS production function for carded cotton textiles was derived as part of a pilot study in the use of direct technical data: technical handbooks and textbooks, machinery catalogs, and consultation with textile engineers. Because of special sampling problems arising in the derivation of production functions from economic time series, some such information must be used in the classification of industries and the choice of relevant input and output dimensions and function types. In this study, the numerical values of the parameters were also estimated from technical data.

A set of three functions describing the relationships between the time rates of input of fiber and of output, the time rates of input of power and output, and the stock of machinery and the time rate of output, was determined for each of ten successive stages in the processing of the raw material, from opening through weaving. Since the output of each stage is the input of the next, all input requirements at each stage were related to final output as well as to the "intermediate" outputs of the individual stages.

This production function was defined operationally in terms of current machinery catalogue information and technical experts' opinions of best current practice, and is known as the "best practice" production function. This differs from the production function which would usually be inferred from industry-wide input-output statistics in that the latter would represent an average of the production functions relevant to the various models of equipment of different ages in use in the industry. The best practice coefficients, $\bar{b}_{ij}$, are useful in explaining changes in the values of the "average" coefficients, b_{ij}, with investment and scrappage.

Consider the fixed coefficient case: Let I/T and S/T represent, respectively, proportions of new machines purchased and old machines scrapped, and let b'_{ij} represent average productivity associated with the discarded machines. Then, over any period t_0 to t:

$$b_{ij}(t) = b_{ij}(t_0) + \frac{I}{T}[\bar{b}_{ij} - b_{ij}(t_0)] + \frac{S}{T}[b_{ij}(t_0) - b'_{ij}].$$

This relationship was used to predict changes in output per spindle-hour over the periods 1925–1935 and 1936–1940.

Once the relevant sets of technical parameters are determined, the task of explaining technological change in the average coefficients (i.e., innovation) can be reduced to the more familiar problems of determining capacity requirements and substitution within a framework of given technical alternatives.

An Airframe Production Function, ARMEN ALCHIAN, University of California at Los Angeles.

LET L be hours of direct labor required to fabricate a particular, e.g., the 100th, airframe (of a specific type in a given manufacturing facility) where $N = 1, 2, 3, \cdots, 100, \cdots$ is the serial number of the first, second, third, etc., airframe of the specified type. L is the direct labor (excluding administrative, design, and tooling time) that, regardless of when it was exerted, is, so to speak, congealed in the Nth airframe. $L = aN^b$, where a and b are parameters, is a widely used function in predicting labor needs and costs. In some instances it has been claimed that b has a value of about $-.33$ regardless of type of airframe or manufacturing facility. The rationale of the functional form is discussed, and in general there is nothing inconsistent or implausible in this functional form if it is regarded as a description of observed data rather than as a rationalization of fundamental forces. With World War II data, the alleged homogeneity of the parameters among different airframes is tested and found inadmissable statistically as well as in practical import. Its goodness of fit as a past summary statistic is good, but its extrapolative ability is poorer. Its predictive reliability as measured by the discrepancy between predicted total direct labor requirements for N planes, $\int L dN$, and the recorded labor requirements is such that the discrepancy averages about 25% of actual requirements with a range of between -40% and $+50\%$. The practical significance of such reliability is discussed.

Hog Slaughtering Production and Cost Functions, DONALD M. FORT and O. H. BROWNLEE, The University of Chicago.

THE PURPOSE of the present study is to estimate some of the effects of hog supply fluctuations on costs of killing, cutting, and lard-rendering for a specific meat packing plant.

Costs are divided into two categories: (a) labor costs of "bonus" gangs (defined below) in killing and cutting, and (b) all other costs. Costs under (a) are tied to output by an established formula involving a modified piece rate system with a minimum weekly pay guarantee and overtime provisions. A choice of one of three alternative killing gangs and of one of four cutting gangs may be made, the alternative gangs differing in size, organization, and output rate. Costs under (b) include variable costs such as labor paid on an hourly basis, supplies, maintenance, and power, and costs which were assumed to be fixed, such as insurance, taxes, depreciation, and administration. The variable costs are assumed to vary linearly, and in certain cases proportionately, with output.

Cost relationships under (a) are computed from the specifications for the pay formula for each gang. Costs relationships under (b) are estimated by simple regression of each cost item on output, based on monthly accounting data supplied by the firm and subject to a priori restrictions that some of the costs be proportional to output, that some others be fixed, and that still others vary linearly but not necessarily proportionately with output.

Hog receipts are subject to daily, weekly, seasonal, and annual fluctuations which are to some extent unpredictable. By the assumption of linearity, costs under (b) during a given time interval depend only on total receipts during that interval, regardless of the degree of fluctuation within the interval. Costs under (a) do depend upon the degree of fluctuation because of nonlinearities resulting from the minimum weekly pay guarantee and overtime provisions and because of the possibility of shifting from one gang to another. These costs thus depend on the time pattern of hog receipts and the time pattern of use of the various gangs. The cost of a given average output can be lowered by reducing the amplitude of supply fluctuations and by increasing the extent to which the least-cost gang is used at each point in time, i.e., by using small gangs for low outputs and large gangs for large outputs.

The present study covers only a portion of processing costs, ignoring, for example, the costs of storage which may depend strongly on the extent of supply fluctuations. Costs of shifting workers between jobs within the plant also have not been estimated. The "fixed" costs apply only to the plant in question. A firm facing a new supply situation could in the long run adjust its size and layout to reduce costs below those estimated here. In addition to the incompleteness of the estimates, errors may have arisen because of misinterpretation of the accounting data or through incorrect a priori assumptions as to the forms of cost relationships.

DISCUSSION by KENNETH MAY, Carleton College—The accepted "theory of production" is rather a theory of imputation whose concepts, assumptions, and conclusions appear to be without analogues in engineering and business experi-

ence. It remains mostly on an abstract, nonoperational level. Its applications have been largely normative and disculpatory. The new theory of production whose beginnings are suggested by this session and recent work on linear models is inspired by the need for quantitative prediction and planning in private industry, war preparations, and the national economy. It replaces the unrealistic assumptions of fixed technology, divisibility, substitutability, and independence by the recognition of ubiquitous technological change, discreteness, capacity, limited substitutability, and constant production coefficients in the short run. The new approach is corrosive of marginal theory (in spite of possible formal reconciliations), but its practical successes hint at a rapid development. This would imply a healthy *rapprochement* between economics and engineering and significant changes in the production side of economic theory.

DISCUSSION by CARL N. KLAHR, Carnegie Institute of Technology and Cowles Commission for Research in Economics, The University of Chicago—Our interest in production functions is two-sided: First we wish to set up models that describe the production phase of economic activity in various industries in order to provide a general model for our economy as a whole. Second, the relative simplicity and large degree of control of production as compared with other economic problems makes it likely that with improving statistical and mathematical models, coupled with an increasing engineering orientation on the part of economists, the production function will be the first economic application to give practical quantitative results. This can lead to a new and exact set of instruments for business planning and control for the individual firm. The most general production function of this type is $b = Od$, where b is bill of goods vector, O is a general operator, and d is a vector of decisions, e.g., budget allocations. In order to calculate such production functions we should rely not only on statistics of inputs and outputs but to a greater extent on an engineering familiarity with the plant and technology being described.

DISCUSSION by HAROLD J. BARNETT, U. S. Department of the Interior—All three papers are welcome contributions of much needed economic data. Mr. Alchian's significant relationship is duplicated throughout the economy of the United States. For the period 1939–1944, for example, labor input per unit of deflated GNP plotted against cumulative deflated GNP yields a "minus 0.27" curve, comparable to the "minus 0.33" curve in airframe production. The historical fact for our society is that technological change hurtles costs downward faster than alleged diseconomies of scale or pressure on resources push them upward, contrary to classical assumptions of increasing average cost. I suggest that Mrs. Grosse's coefficients, derived from handbooks, are not as independent of average observed data as she implies. I doubt, therefore, that these sources yield technological horizons which can be viewed in innovation studies as moving asymptotes, which average technologies approach. Mr. Fort's data demonstrate, contrary to an early premise in his paper, that linearity is probably not a good assumption for this plant. For example, to achieve linearity in labor input, he would have to assume that laborers in identical functions are not a single class of inputs and that maintenance costs are negative at nil output. I suggest that assumptions of linearity should always be related to specific, narrow output ranges.

DISCUSSION by ERNEST C. OLSON, Board of Governors of the Federal Reserve System—The linearity of the logarithmic production function reported by Mr. Alchian appears to be more characteristic of the data at production levels below

700 airframes than it is for the sample as a whole. At higher levels, ranging from 700 to 2000 airframes, a rather sharp departure from linearity is apparent. Whether this is significant remains to be established by further investigations, but the data presented appear to indicate that production in this range was associated with a smaller labor outlay per unit of airframe produced than is indicated by the regression equation. In terms of the general significance of the results reported, it should be noted that the data refer to wartime conditions and, as Alchian points out, undoubtedly reflect the learning experience of a labor force characterized by a high rate of turnover. With a lower rate of turnover under peacetime conditions of production, it is quite possible that a set of parameters would be found which differ significantly from those reported.

Mrs. Grosse presented an impressive study of the textile production function for cotton yarns and cloth. The use of technical industrial data to obtain the parameters of this function represents an interesting departure from the more conventional procedures of statistical induction and is a relatively new approach to the problem. Since such technical information is directly related to actual industrial processes, its use in the determination of the parameters of the function appears to have made possible a degree of precision not otherwise obtainable.

NOTICE OF THE TOKYO MEETING

July 7–10, 1950

The first regional meeting of the Econometric Society in Japan will be held in Tokyo this summer concurrently with the meeting of the Japanese Statistical Association. The tentative dates for the meeting are July 7–10, 1950. A report of the Tokyo meeting will appear in a subsequent issue of ECONOMETRICA.

NOTICE OF THE BERKELEY MEETING

August 1–5, 1950

The western regional meeting of the Econometric Society will be held August 1–5, 1950, on the Berkeley campus of the University of California in conjunction with the Second Symposium on Mathematical Statistics and Probability which extends from July 31 to August 12. Details as to the program and arrangements for attendance are contained in an announcement mailed to American members in June. Those wishing copies of that announcement or additional information should write to the Econometric Society, The University of Chicago, Chicago 37, Illinois, or to either Professor Kenneth J. Arrow, Department of Economics, Stanford University, Stanford, California (program chairman), or Professor George Kuznets, Giannini Hall, University of California, Berkeley 4, California (local arrangements representative).

NOTICE OF THE HARVARD MEETING

August 31–September 5, 1950

In place of the summer meeting which the Econometric Society customarily holds in conjunction with the mathematical associations, the Econometric Society will meet this year at Harvard University, Cambridge, Massachusetts, August 31–September 5, concurrently with the International Congress of Mathematicians which meets at Harvard, August 30–September 6.

In addition to a number of sessions devoted to contributed papers, sessions have been organized on the following topics: The Determination of Production Functions, Scientific Method, Application of Statistical Method to the Solution of Economic Problems, Activity Analysis in the Theory of Production, Utility Measurement, National Accounting and Measurement of Production, and Consumer Behavior. A number of sessions of the Congress will also be of particular interest to members of the Econometric Society.

Details as to the program and arrangements for attendance are contained in an announcement mailed to American members in June, copies of which are available on request from the Econometric Society, The University of Chicago, Chicago 37, Illinois. Those planning to attend the Harvard meeting who have not availed themselves of the housing arrangements described in that announcement should place their reservations directly with the Commander Hotel, Cambridge, Massachusetts, or with other hotels in the Cambridge and Boston area. Inquiries regarding the presentation of papers at the Harvard meeting should be addressed directly to the program chairman, Professor Arthur Smithies, Department of Economics, Harvard University, Cambridge 38, Massachusetts.

NOTICE OF THE VARESE MEETING

September 6–8, 1950

The dates for the twelfth European meeting of the Econometric Society to be held in Varese, Italy, this year have now been set as September 6–8, inclusive. This immediately follows the XIVth International Congress of Sociology in Rome (inquiries regarding which should be addressed to Dr. Corrado Gini, c/o Societa Italiana di Sociologia, via delle Terme di Diocleziano, N. 10, Rome) and precedes the Round Table Conference of the International Economic Association which is scheduled for September 10–13 at Monaco.

Professor Felice Vinci, Via Lamarmora 42, Milano, Italy, has accepted appointment as chairman of the program committee for the Varese meeting, and Professor Eraldo Fossati has undertaken the responsibility for local arrangements. The address for Professor Fossati after July 1 will be c/o Ente Provinciale del Turismo, piazza Monte Grappa, Varese, Italy. Other members of the program committee include Professor R. G. D. Allen, London School of Economics; Professor Ragnar Frisch, University of Oslo; and Professor René Roy, Paris, France.

Members interested in participating in the Varese meeting are encouraged to contact the program committee for further information.

NOTICE OF THE CHICAGO MEETING

December 27–31, 1950

The American winter meeting of the Econometric Society will be held in Chicago, Illinois, Wednesday, December 27, to Saturday, December 30, 1950. Sessions of primary interest to economists and those which are sponsored jointly with the American Economic Association and Ameri-

can Farm Economic Association will be held at the Palmer House throughout the period. Sessions which, in addition, are of interest to statisticians will be held during the first three days of the period at the Congress Hotel and Roosevelt College, in conjunction with the meetings of the American Statistical Association and Institute of Mathematical Statistics.

Sessions on the following topics are at various states of preparation by the program committee of the Econometric Society or co-sponsoring organization: Theory of Comparative Advantage and Patterns of World Trade, Utility Analysis of Decisions Involving Risk, Business Expectations in Business Planning, History of Mathematical Economics, Current Input-Output Studies, Welfare Economics, Problems of Incorrect and Incomplete Specification, Econometric Methods in Agricultural Research, Collection and Use of Survey Data, Demand Analysis, Computation Problems in Theory of Games, Multivariate Analysis, Time Series, and Recent Advances in the Theory of Decision Functions. A number of sessions for contributed papers will also be provided.

Abstracts of papers to be considered for the December meeting should be sent prior to September 15, 1950, to Professor Clifford Hildreth, c/o The Econometric Society, The University of Chicago, Chicago 37, Illinois. Other members of the program committee are listed in the April issue of ECONOMETRICA. Further details will be announced as plans progress.

ERRATA

The following corrections in the manuscript of "Rational Behavior, Uncertain Prospects, and Measurable Utility" by Jacob Marschak (April, 1950) have been communicated by the author. Page 123, line 1: for a read a'; line 2: for $<$ read $\leq$.

ECONOMETRICA

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AN ELECTRO-ANALOG METHOD FOR INVESTIGATING PROBLEMS IN ECONOMIC DYNAMICS: INVENTORY OSCILLATIONS

BY N. F. MOREHOUSE, R. H. STROTZ, AND S. J. HORWITZ[1]

This paper discusses the possible use of the Aeracom-type analog-computer in solving dynamic economic models. The first part presents an industry inventory model as an illustration and shows solutions that were obtained by use of the Aeracom for various parameter values. The latter part discusses the types of equations that can be handled by the Aeracom and the general applicability of the computer to economic models.

ALTHOUGH analogy with the physical sciences has characterized economic thought since its origins, the purpose in constructing such parallelisms has remained entirely expository. With the development in recent years of advanced mathematical computers based upon electrical circuits, however, economists ought to inquire seriously into the aid their subject might receive from the investigation of and actual experimentation with electrical analogs.[2]

Two types and uses of the modern computational devices may be distinguished. First, there are the digital computers (for example, the IBM computer at Harvard) which may sizably reduce the computational burdens inherent in modern statistical methods. Professor Frisch has written quite optimistically regarding these possibilities.[3] Secondly, there are the analog-computers, one type of which is the Aeracom at Northwestern University. These machines are peculiarly well adapted to the study of oscillations. It is the purpose of this paper to consider some of

[1] The authors wish to acknowledge the kind cooperation of Dr. J. F. Calvert, Mr. J C. McAnulty, and Miss Marjorie Wiley, of the Aerial Measurements Laboratory, Northwestern University.

[2] It is of historical interest to note Irving Fisher's exceptional claim for the hydraulic analogies he employed: "We are thus enabled not only to obtain a clear and analytical *picture* of the interdependence of the many elements in the causation of prices, but also to employ the mechanism as an instrument of investigation and by it, study some complicated variations which could scarcely be successfully followed without its aid." [*Mathematical Investigations in the Theory of Value and Prices*, New Haven: Yale University Press, 1925 (first published in 1892), p 44.]

[3] Ragnar Frisch, "Repercussion Studies at Oslo," *American Economic Review*, Vol. 38, June, 1948, pp. 367–372.

the potentialities of the analog-computer in the investigation of economic dynamics.

The Aeracom is a room-sized machine designed for the ready construction of various electrical circuits. Once a particular circuit has been connected, variation in one (or more) of the electrical magnitudes (e.g., voltage, resistance, capacitance, etc.) causes variation in the other variables of the system. The effect on each of the other (dependent) variables may then be seen on an oscilloscope screen as a time series and recorded photographically. Changes in the parameter values are easily made by turning dials. Until the present time, the Aeracom has found its main employment in the study of various physical and engineering problems. The mathematical representation of mechanical, acoustic,

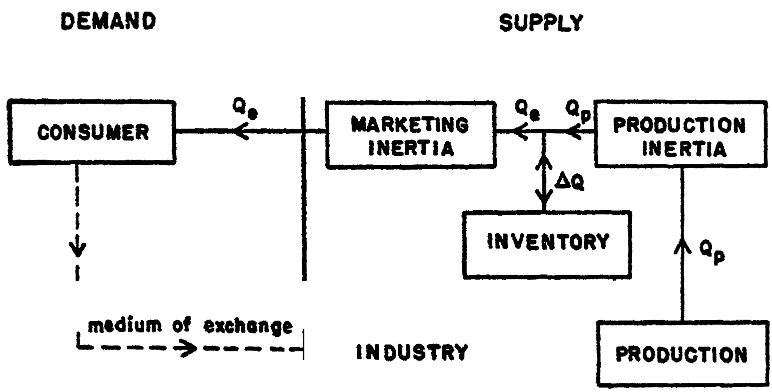

FIGURE 1—Economic model.

and hydraulic systems may often be found to correspond to the mathematical representation of contrived electrical circuits. Hence, by analogy, the mathematical solution of the electrical system is the same as that of the mechanical, acoustic, or hydraulic system under investigation. It is the thesis of this paper that electrical analogs may also be constructed for economic models and the properties of the models investigated in a similar way.

I. AN INVENTORY MODEL

In this article a simple economic model and its electrical analog are constructed with the main purpose of illustrating the applicability of the electrical-analog method. Variations in the parameters are then introduced and their consequences examined by use of the Aeracom. This work has been conducted in the Aerial Measurements Laboratory of the

Technological Institute at Northwestern University. The Aeracom was made available through the cooperation of the Navy Department, Bureau of Aeronautics.

The model to be investigated consists of a simple supply and demand situation, Figure 1, in which during equilibrium goods are being supplied by an industry to consumers at a fixed rate and are being produced by the industry at an equal rate, so that inventory remains at some constant level. The problem is to discover the disequilibrium behavior of the system in response to a sudden increase in consumer demand, assuming now a long-run management policy of rebuilding the inventory exponentially to its original equilibrium level. The increase in demand, however, is assumed not to exhaust the inventory at any time. In order to make the problem more interesting, response lags are introduced into the production and marketing departments, and the resulting effect of demand variation on the inventory position of the industry is studied. Throughout this paper these "response lags" will be called "inertias."

We begin by postulating a linear demand function

$$P_d = \alpha_1 - \beta_1 Q_e , \tag{1}$$

and a linear output function

$$P_s = \alpha_2 + \beta_2 Q_p . \tag{2}$$

Q_e is the quantity exchanged, and P_d is the demand price for this quantity. Q_p is the quantity produced, and P_s is the lowest price at which the industry will produce this quantity. $Q_p - Q_e$ is an inventory increment. While these quantities are really rates, they need not here be treated explicitly as such. The industry is assumed to seek a certain level of inventories, Q_i , and short-run deviations from this level result from the dynamic elements of the market.

In equilibrium, $P_d = P_s =$ market price, $Q_e = Q_p$ (with Q_i constant), and the output function may therefore be regarded as a "normal" supply function. If it be assumed that the industry charges the highest price that it can get for its current rate of sales, market price is always P_d .

Suppose that from an initial equilibrium position (P^0, Q^0) disequilibrium is introduced by a rise in α_1 at time T^0. To explain the time path to the new equilibrium position (P^1, Q^1) attained at time T^1, two further functions are required. The nature of these functions must depend upon the properties of the market which are postulated. We choose to consider two linear time lags. The industry is assumed to respond to the sudden increase in demand both by gradually unloading its inventories and by gradually increasing its production so as to rebuild its inventories and to keep up with the greater rate of sales. The industry may overshoot its

new equilibrium level, however, because of lack of knowledge of the new equilibrium position. For example, its response may be initially to meet the increased demand more by reducing inventories than by increasing production. Subsequent production increases, however, may gain momentum and push beyond the new equilibrium level. Inventories may possibly rise above Q_i, and production must then decline, not simply because it is no longer necessary to increase inventories, but because inventories must actually be reduced. Oscillations are thus introduced by a familiar accelerator principle combined with an "overcompensation" principle. This behavior is described by the following equations:

$$P_d - P_s = \lambda_1 \dot{Q}_s + \lambda_2 \dot{Q}_p \tag{3}$$

and

$$P_d - P^0 = \lambda_1 \dot{Q}_s + \frac{1}{\gamma} \int_{T^0}^{T} (\Delta Q)\, dT + P_i, \tag{4}$$

where $\Delta Q = Q_s - Q_p$, T is time, P_i is an inventory control function, to be explained somewhat later, and the dot indicates a derivative with respect to time.

A major problem in the formulation of dynamic economic models is the derivation of these adjustment equations. Various approaches might be followed, but it is not the purpose of this paper to discuss their relative merits. We choose, for illustrative purposes, to adhere to the familiar tradition of employing differential equations.

In our particular problem we may think of the excess of demand price over supply price as an incentive for an increase in production. If demand price rises, this incentive has increased and, in the absence of any sluggishness of response (or inertias) of the marketing or production departments, sales must immediately increase until demand and supply prices are once again equal. In the model under consideration there are, however, two inertia elements introduced: (1) the slowness of response of the marketing departments in increasing their sales either by drawing from inventories or from increased production; and (2) the slowness of the production departments to step up the rate of output.[4] These inertias

[4] The first inertia or time lag may result either from the slowness of the firm to realize that its demand position has improved and/or from technical obstacles in the marketing process. Similarly, the second lag may be attributable to psychological and/or technical factors. In the text, inventories are regarded as made up of finished goods. However, if inventories take the form of semifinished goods, there would also be time lags involved in either adding to or subtracting from inventories. As the reader proceeds through this paper he may appreciate that an entire series of inventories involving goods in various stages of production and the associated time lags could readily be introduced.

are introduced as counterincentives which operate against changes in the rates of sales or of production and are represented by the symbols λ_1 and λ_2. These appear as coefficients of the rates of change in quantities sold or produced, respectively. Thus, as equation (3) tells us, the incentive of the excess of demand price over supply price working towards greater sales and output is counterbalanced by the rates of change in sales and output, each multiplied by the respective inertia coefficients.

The second adjustment equation relates demand price to the condition of inventories. We regard the excess of demand price over the initial equilibrium price P^0 as an incentive for increasing the exchange of goods which is offset by the inertia of the marketing departments and the state of the inventory. The inertia of the marketing departments is $\lambda_1\dot{Q}_e$, as before. The condition of the inventory is expressed by $(1/\gamma)\int_{T^0}^{T}(\Delta Q)dT + P_t$, where $\Delta Q = (Q_e - Q_p)$. The term containing the integral may be said to represent the reluctance of the industry to permit its actual level of inventories to fall beneath (or, if negative, to exceed) the equilibrium level of inventories. The coefficient $1/\gamma$ may be said to represent the degree of inventory inflexibility. The term P_t is introduced to control the new equilibrium level of inventory and the manner in which the new level will be approached. If $P_t = 0, \int_{T^0}^{T^1}(\Delta Q)dT = \gamma(P^1 - P^0) > 0$. This is so because the rise in α_1, by raising equilibrium price, causes the inventory warehouses to wish to hold smaller inventories.[5] Taken alone, this clearly is not an irrational relationship. As the value of the commodity increases, it becomes more costly for the firm to tie up any given amount in inventories. However, there are other considerations which help determine long-run inventory policy, e.g., price expectations, volume of sales, etc. Consequently, as a result of the change in the equilibrium position, what may be called the "reservation demand" function of the firm may shift. Unspecified is the pattern of shift in this function through time. Here we felt the best simple assumption we could make is that this shift occurs exponentially and suffices to establish the new equilibrium level of inventories equal to the original equilibrium level. This is accomplished by setting $P_t = (P^1 - P^0)(1 - e^{-\mu(T-T^0)})$. This assures us that, although a variation in price level will produce changes in inventory level, there will be superimposed upon these effects a gradual rebuilding in an exponential fashion until, after a certain length of time determined by the value of μ, the inventory level will be essentially the same as before. By employing a photoformer (discussed in Section 2) or other nonlinear circuit elements, a broader variety of long-run inven-

[5] It is therefore not exactly proper to define $(1/\gamma)\int_{T^0}^{T}(\Delta Q)dT$ as the industry's "reluctance" to reduce inventories. This reluctance is more properly defined as $(1/\gamma)\int_{T^0}^{T}(\Delta Q)dT - \gamma(P^1 - P^0) + P_t$.

tory policies may be given electrical representation. We now rewrite (4) as

$$(4') \qquad P_d - P^0 = \lambda_1 \dot{Q}_e + \frac{1}{\gamma}\int_{T^0}^{T} (\Delta Q)dT + (P^1 - P^0)(1 - e^{-\mu(T-T^0)}).$$

Another adjustment equation, relating supply price to the condition of inventory, can be written as

$$(5) \qquad P_s - P^0 = -\lambda_2 \dot{Q}_p + \frac{1}{\gamma}\int_{T^0}^{T} (\Delta Q)dT + P_i\,;$$

but this equation is redundant since it can be derived by subtracting equation (3) from equation (4).

As a check on our thinking, let us reduce the adjustment equations to the simplified case of equilibrium. To think of the system in the initial equilibrium state at $T = T^0$, we make $Q_e = Q_P = Q^0$ and set all time derivatives equal to zero. Equation (3) then reduces to $P_d = P_s$, which is the expected equilibrium condition. Equation (4′) reduces to $P_d = P^0$ if we note that $\int_{T^0}^{T} (\Delta Q)dT$ will be zero and $e^{-\mu(T-T^0)}$ will be 1 at $T = T^0$. When we think of the final equilibrium state, we make $Q_e = Q_p = Q^1$. Equation (3) reduces to $P_d = P_s$, as before, and equation (4′) becomes essentially $\int_{T^0}^{T^1} (\Delta Q)dT = 0$ for a sufficiently large value of T.

Since the model is now completely defined economically, we can proceed to express it in electrical terms. Making use of an analogy between the economic concepts such as flow of goods, incentives, inertias, and inventory and the electrical equivalents such as current, voltage, inductance, and capacitance, we may recast the economic equations (1) through (5) into the corresponding electrical circuit equations (6) through (10) below. Note that the expressions correspond term for term. We have

$$(6) \qquad E_1 = V_1 - R_1 I_1\,,$$

$$(7) \qquad E_2 = V_2 + R_2 I_2\,,$$

$$(8) \qquad E_1 - E_2 = L_1 \dot{I}_1 + L_2 \dot{I}_2\,,$$

$$(9) \qquad E_1 - E^0 = L_1 \dot{I}_1 + \frac{1}{C}\int_{t^0}^{t} (I_1 - I_2)\,dt + E_i\,,$$

$$(10) \qquad E_2 - E^0 = -L_2 \dot{I}_2 + \frac{1}{C}\int_{t^0}^{t} (I_1 - I_2)\,dt + E_i\,,$$

where $E_i = (E^1 - E^0)(1 - e^{-(R_2/L_2)(t-t^0)})$.

For the electrical circuit, R is resistance, L is inductance, C is capacitance, V is battery voltage, E is voltage across one or more circuit ele-

ments, I is current, q^0 is initial charge on C, D is a diode rectifier permitting current flow only in the direction indicated, and t is time.

From the circuit equations (6) through (10) and a knowledge of Kirchhoff's laws governing voltages and currents of networks, a circuit of electrical elements may be devised whose behavior is defined by the equations. This circuit is shown in Figure 2. The purpose of the diode rectifiers[6] D_1 and D_2 is to restrict the current flow to one direction to match the unidirectional flow of goods dictated by economic considerations. This means that consumers do not sell back to the firms, nor do the firms convert inventories back into the factors of production; that is, sales and production cannot become negative. The battery voltage V_1 must

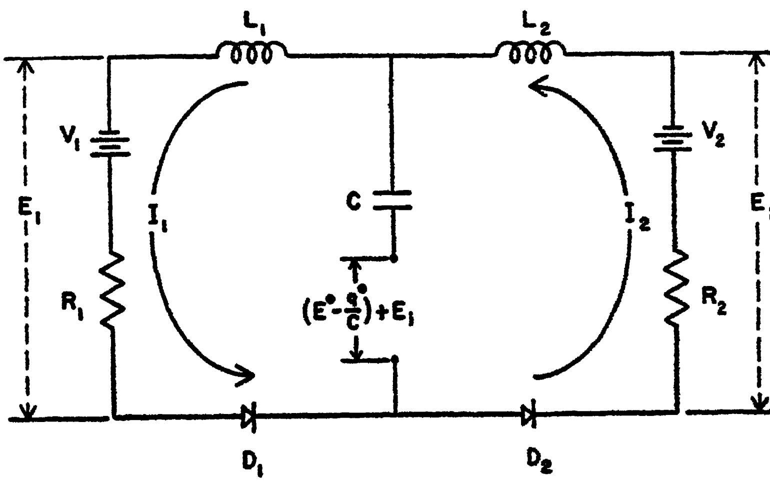

FIGURE 2—Electrical circuit.

be greater than V_2 in order to have a current flow at all. In economic terms, this is the condition that the demand and supply curves intersect in the positive quadrant. Across the portion of the circuit below the capacitor C is impressed a voltage which serves a dual purpose. Part of this voltage keeps the capacitor charged positively to match the economic concept of a level of inventory which always remains positive, and the remainder is used to rebuild the charge on the capacitor to match the economic postulate of a long-run constant inventory level.[7]

The circuit of Figure 2 was plugged into the Aeracom, and the values of the elements were set to simulate a reference equilibrium position of

[6] Each diode rectifier has a certain resistance which is to be regarded as included in the measure of the neighboring resistance.

[7] The auxiliary circuit for producing $E^0 - (q^0/C) + E_1$ is described in the Appendix.

the model. Then the equilibrium of the circuit was disturbed by suddenly increasing the value of the voltage V_1, and the solution (the behavior of some selected variable) was observed as a time function on the oscilloscope screen. After several different variables had been selected in turn, the values of the elements were rapidly changed and solutions for the new conditions were obtained and recorded photographically.

These photographs are the final stage of the Aeracom work. Their interpretation in economic terms should furnish the desired information about the behavior of the given model. In order to demonstrate the method of interpretation, we have selected for presentation a few of the many photographs taken during the study and shall discuss them in the concluding pages of this section. We have chosen to show photographs of the curves Q_e, Q_p, ΔQ, and $\int_{T^0}^{T} (\Delta Q)dT$ versus time. Each photograph shows, for one of these quantities, the initial equilibrium position, the disequilibrium behavior introduced at T^0 by a rise in α_1, and finally the new equilibrium position.

The photographs illustrating these points are shown in Figures 3 through 8. Figure 3 was taken as a general reference case with which the other cases could be compared. The ordinate scales of the top three photographs in each figure represent flow of goods. For the picture of ΔQ, values above zero mean flow out of inventory and values below zero mean flow into inventory. The ordinate scale of the bottom picture differs from the others in that it represents total amount of goods rather than flow. In each picture we see at the left a straight line which is the constant value for the initial equilibrium state. A sharp break introduces the adjustment behavior, which is followed by the leveling-off of the curves at final equilibrium. Each column of photographs shows the changes which occur when one or two of the parameters are altered from their values in the reference case.[8] The cases are described below.

Figure 3. For this general reference case the value of the intercept parameter α_1 was suddenly increased at T^0 by half its value. The other intercept parameter, α_2, was held at zero.[9] The slope coefficients β_1 and β_2 were set at equal arbitrary values, as were the inertia coefficients λ_1 and λ_2. The inventory coefficient $1/\gamma$ was set arbitrarily at a value which would produce a definite oscillation. The inventory control coefficient, μ, was adjusted so that the inventory was very nearly back to its original level within the picture space.[10]

[8] The values of the electrical elements used in the study are given in the Appendix

[9] This is not a special case; it means merely that the voltage level of the system has been shifted in order to eliminate the battery V_2 from the circuit

[10] Theoretically, the new equilibrium is reached only after an infinite length of time.

The figure shows clearly that the industry responds first by taking goods out of inventory rather than by increasing production because ΔQ, the flow out of inventory, initially increases faster than production,

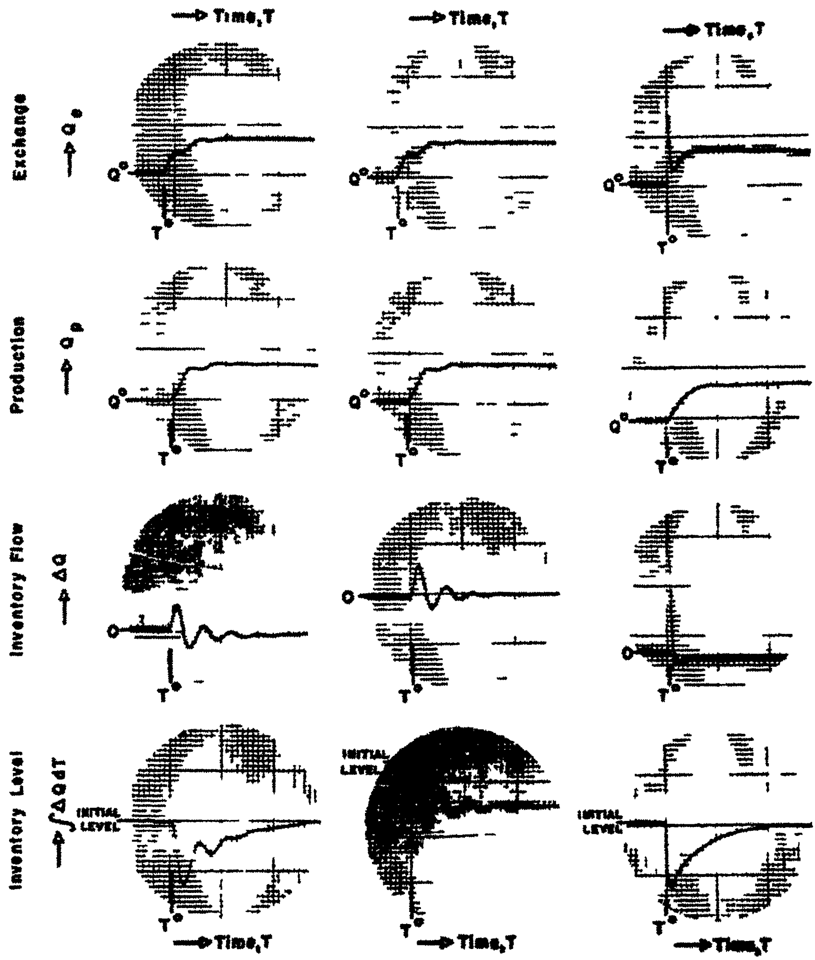

Figure 3—General reference situation

Figure 4—Inventory level not restored

Figure 5—Marketing inertia removed

Q_p. This is due to the fact that the effect of production inertia exceeds the marketing inertia. The drain on the inventory depletes it to the level justified by the new demand price, at which time goods should stop flowing out of inventory. Because of marketing inertia, however, goods

continue to flow out of inventory below this point. The desire of the industry to keep inventory at suitable levels then comes into play to divert part of the production from the consumer (note the flat spot on

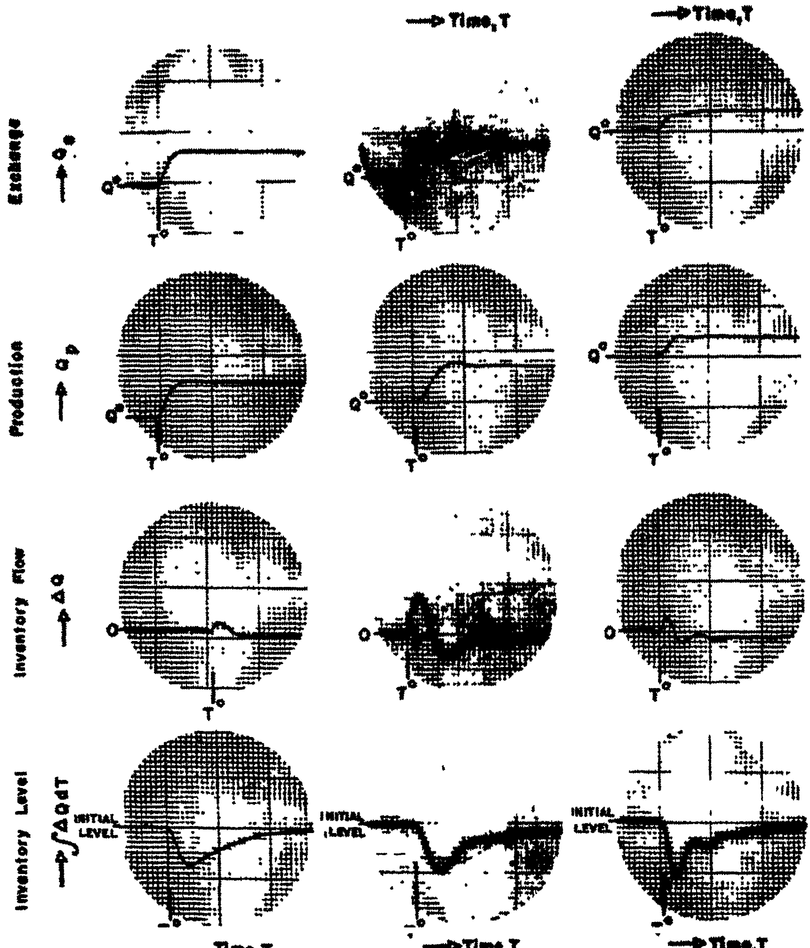

FIGURE 6—Production inertia removed.

FIGURE 7—Inventory coefficient decreased.

FIGURE 8—Slope coefficients increased.

Q_e) to inventory in order to halt the overdepletion of the inventory and bring it back up to the new desired level. Here, however, the production inertia operates to allow the inventory to increase too much. This means the inventory incentive must again act to reverse the flow, or to divert

production from inventory to the consumer, thus momentarily effecting an apparent overproduction which induces a production slowdown (note the flat spot on Q_p). The cycle begins again as inventory now decreases and for the same reasons as before goes below its proper level, but not so far this time. The inventory level oscillates in this fashion before settling down; and Q_e, Q_p, and ΔQ reflect these oscillations. The oscillations represent the inability of the industry to foresee changes and to react instantly once the changes are apparent.

Figure 4. The effect on the short-run industry oscillations of the desire to maintain a long-run constant inventory may be seen by eliminating this desire; this is shown in Figure 4. Note that the inventory fluctuates about a new and lower constant level but that the oscillations in these pictures show little change from those in the corresponding pictures in Figure 3.

Figure 5. Here we have the response when there is no marketing inertia, other things remaining the same. The curves of Q_e and ΔQ rise immediately; i.e., all exchange comes from inventory at first. But the inventory drops slightly below the point justified by the new demand price, and therefore goods stop flowing out of inventory (ΔQ drops to zero) and Q_e drops. Meanwhile, production is increasing to the point where it takes care of both the quantity exchanged and the quantity needed for inventory. A minus value of ΔQ indicates a flow of goods into inventory, gradually rebuilding the inventory level.

Figure 6. These pictures show the response when the production inertia is set to zero, the other parameters being the same as for Figure 3. Here marketing inertia acts equally on inventory and production. The reaction is slow, Q_e being made up initially of about equal parts of production increase and rate of inventory depletion. Production gradually takes over the new rate of exchange and slowly rebuilds the inventory. This, of course, represents the behavior of an industry whose inventory policy is independent of the production inertia.

Figure 7. The effect of reducing the inventory coefficient $1/\gamma$ to one quarter of its value in Figure 3, the other parameters remaining the same, is shown in this figure. Here the vertical scale of $\int_{T_0}^{T} (\Delta Q)dT$ is compressed four times. A reduction of $1/\gamma$ means that the industry is willing to allow the inventory to be depleted more before rebuilding to the initial point of equilibrium. Examination of the photographs shows that the time required for one oscillation has about doubled. The oscillations are generally mild.

Figure 8. The final figure shows the response when the production and consumption coefficients β_1 and β_2 are doubled, the other parameters remaining the same as in Figure 3. The industry still shows the same oscillations as before, but the increases in production and exchange are

only a little more than one half what they were in Figure 3, for the same change in demand. This indicates a general resistance to change in the system.

Thus we see that the dynamic behavior of this economic model under a variety of conditions can be readily studied by observing the electrical analog. A particular advantage of the electrical-analog method is the ease with which the element to be observed can be changed, the parameters of the model varied, or the model itself altered. The method should prove an adaptable and convenient tool for the exploration of increasingly complex economic models.

II. GENERAL APPLICABILITY OF ELECTRO-ANALOG METHOD

The preceding model falls short of illustrating the full versatility of the electro-analog method in economics. Neither an economic system nor the Aeracom is restricted to integro-differential equations. In the remainder of this article other mathematical features of economic models and the electro-analog method are discussed. It should be emphasized, however, that this discussion is concerned with prospects rather than with past experiments, and many of the claims for the electro-analog method which are made "in principle" are made on the assumption that certain engineering problems can be solved.

Exogenous variables. Exogenous variables which are arbitrary functions of time may be introduced in an analog as voltages, capacitances, or inductances by use of a *photoformer*.[11] To use this device one cuts out an opaque mask whose profile is any arbitrary single-valued function of time. This mask, placed in the photoformer, then causes a voltage to vary through time in accordance with the profile. Capacitances and inductances may also be made to vary through time by making them functions of such a voltage. Photoformers have not yet been employed by the Aerial Measurements Laboratory. It is planned to use a photoformer during the next few months, however, in an illustrative national income model.

Difference equations. Difference equations may be included in an analog by the use of a "memory" unit. This is a magnetic tape on which a variable determined by the circuit is recorded at one time point and played back into the circuit after a finite time lag. The length of the time lag may be controlled arbitrarily. There is in principle no limit to the number of lagged values that may appear in any single equation or in a given model. Furthermore, lagged values of integrals or of differentials may also be included. The authors have plans at present to construct on

[11] For a description of the photoformer, see H. W. Schultz, J. F. Calvert, and E. L. Buell, "The Photoformer in Anacom Calculations," *Proceedings of the National Electronics Conference*, Vol. 5, 1949, pp. 40–47.

the Aeracom a national income model illustrating the use of a difference equation.

Stochastic variables. It is believed that stochastic variables may be introduced by superimposing on any driving function (voltage) a random element obtained by playing into the system a sample of atmospheric "noise" as recorded on a magnetic tape. It is felt that such noise has a normal distribution, the variance of which may be altered by varying the amplitude of the "play-back." These disturbances appear as a continuous function of time so that there is serial correlation for a sufficiently small time lag. This time lag may be made such a small fraction of the entire period of oscillation, however, that the existence of serial correlation can be ignored. If it is desired to include serial correlation of the random disturbances, this may, it is hoped, be achieved by "stretching out" the noise function horizontally by reducing the speed with which the noise is played into the circuit. It should also be possible, in principle, to make both the standard deviation and serial correlation of the random disturbances depend upon the value of any particular variable in the system or upon time. There are, however, engineering problems which remain to be solved; ideally, a special "stochastic unit" for use in economic analogs should be constructed.

An attempt was made to introduce a stochastic variable into the demand function of the inventory problem discussed above. Figure 9 shows a portion of one sample of random disturbances with mean zero. Figure 10 shows the effect of superimposing these random disturbances on V_1 when $V_1 = 10$ volts, $V_2 = 0$ volts, $R_1 = R_2 = 600$ ohms, $L_1 = L_2 = 3.0$ henries, and $C = 1.0$ microfarads. As Figure 10 shows, the Slutsky[12] effect was quite prominent, especially in the behavior of Q_p. It is to be noted that the horizontal scale of Figure 9 is six times greater than the horizontal scale of Figure 10.

It is known that the noise recording used was not played into the circuit with satisfactory fidelity, however. Two samples of the noise as received by the circuit were fitted with normal curves; the χ^2 test for goodness of fit gave values for P of 0.15 and 0.25. Further experimentation with stochastic models is in progress.

Multiplication and partial derivatives. The problem of multiplying two variables is still largely unsolved at the practical level, although theoretical principles for such multiplication have been worked out. The Aeracom is not suited to systems containing partial derivatives.

Nonlinearities. There are various ways in which nonlinearities may be introduced in an analog. First, the photoformer may be employed for nonlinear functions where the argument may be either time or any circuit

[12] Eugen Slutsky, "The Summation of Random Causes As the Source of Cyclic Processes," ECONOMETRICA, Vol. 5, April, 1937, pp. 105–146.

variable. Not only voltages but inductances and capacitances may be made to vary nonlinearly by use of the photoformer. Secondly, nonlinear resistances, inductances, and capacitances may be employed. These consist of a chain of linear elements, different segments being operative at different levels of current. A third method is to employ exponentials introduced by means of a subsidiary R, L circuit, an example of which was presented in the inventory model treated in this paper and explained in the Appendix.

Accuracy. It is not possible to make many general comments regarding the accuracy of the Aeracom. It is clear that analog-type computers do not possess the accuracy of digital computers. The individual passive elements (inductances, resistances, and capacitances) are accurate within

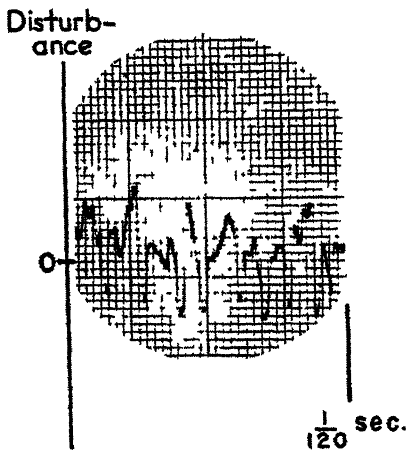

FIGURE 9

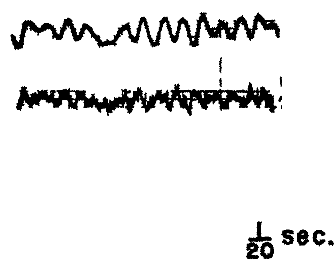

FIGURE 10

limits of one per cent. The limits of accuracy for various circuits will vary, however, depending upon the nature of the circuit. It has been estimated that the cumulative error in a complicated analog may be as much as five per cent. It should be realized that the degree of accuracy obtained depends to a considerable extent upon the care with which the engineering work is performed. Again, depending on the nature of the problem, it may or may not be possible to devise mathematical methods for checking the accuracy of a given solution. Our own experience with the problem discussed in this paper indicates that this particular circuit reproduced very well. It was set up twice, each time different equipment being used. It was not possible to distinguish visually between photographs of the solutions for the two trials.

In any event, the degree of accuracy seems satisfactory for many important economic problems and is certainly adequate to reveal the nature of the dynamic effects caused by variation in parameter values. Much more serious uncertainties are likely to exist in any statistical data with which the economist must work or in the idealizations and approximations introduced in the construction of an economic model. More general statements regarding the accuracy of the computer for dynamic economic models cannot be hoped for, short of the accumulation of much more experience with such models.

Statistical inference.[13] It is hoped that the analog method may eventually be used in the statistical inference of parameters in stochastic models of simultaneous equations. By turning dials and thus varying the structural parameters, theoretical fluctuations of the dependent variables may be compared with historically observed time series. If it is possible to define a "best fit" in some sense, the dial settings that yield the best-fitting theoretical fluctuations are then estimates of the structural parameters. Theoretical problems in statistical inference would be encountered here: How is "best fit" to be defined? What sort of measurements would be required? When is it possible to "zero in" on the best fit by successive approximations? Or, alternatively, how might the dials be turned so that the relevant domain of the parameter values may be adequately searched? Nonetheless, we are hopeful that some answer to these questions can be obtained, if only in a rather arbitrary and inexact way. The parameter estimates that would be obtained might well be inferior to the maximum-likelihood estimates for the same model worked out by modern econometric methods. But the rapidity of the analog method and the facility of effecting minor alterations in a model may mean that the Aeracom would serve well as a *selector* of those models that warrant the application of involved statistical techniques. It may be found, moreover, that the analog method possesses greater versatility than current statistical methods with respect to the sort of model that may be studied. The potential inclusion of stochastic variables whose variances change through time or are related to certain other variables in the model, or whose serial correlations may be controlled, and the inclusion of nonlinear elements may well broaden the vistas of modern econometric research.

[13] This paragraph should be regarded as only a first hint concerning prospects for the analog method in statistical inference. Many problems have been deliberately omitted, and doubtless many others have been omitted quite unknowingly. We also wish to acknowledge that there are many points of contact between the approach considered here and the suggestions contained in Guy H. Orcutt, "A New Regression Analyser," *Journal of the Royal Statistical Society*, Vol. 111, Part I, 1948, pp. 54–70.

APPENDIX

The voltage $(E^0 - q^0/C) + E_t$ is introduced into the inventory leg of the circuit as shown in Figure 11.

The first two terms of the desired voltage are supplied by a battery whose output V_4 is made equal to $E^0 - q^0/C$. The term E_t is the voltage impressed across R_3 by the auxiliary circuit when the switch is closed.

We can write the equation for the I_3 loop as

$$R_3I_3 + R_3(I_1 - I_2) - V_3 + L_3\dot{I}_3 = 0,$$

noting that $I_3 = 0$ at $t = t^0$.

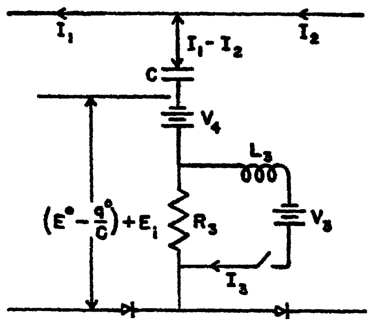

FIGURE 11—Inventory circuit.

If we neglect the mutual impedance term $R_3(I_1 - I_2)$, which may be made as small as we please, we can solve the revised differential equation by standard methods, obtaining

$$I_3 \cong (V_3/R_3)(1 - e^{-(R_3/L_3)(t-t^0)}),$$

or

$$I_3R_3 \cong V_3(1 - e^{-(R_3/L_3)(t-t^0)}).$$

By making V_3 equal to $(E^1 - E^0)$, the voltage across R_3 becomes approximately the desired

$$E_t = (E^1 - E^0)(1 - e^{-(R_3/L_3)(t-t^0)}).$$

In the circuit studied, the values of R_3, L_3, V_3, and V_4 were held constant at 50 ohms, 1.5 henries, 6 volts, and 90 volts, respectively. The values of the other electrical elements are given in the table below for the various cases. The voltage V_1 was changed each time at t^0 from an initial 12 volts to 18 volts. The voltage V_2 was 0.

VALUES OF ELECTRICAL ELEMENTS

Figure	L_1 (henries)	L_2 (henries)	R_1 (ohms)	R_2 (ohms)	C (microfarads)
3	3.0	3.0	550	550	1.0
4	3.0	3.0	550	550	1.0
5	0	3.0	550	550	1.0
6	3.0	0	550	550	1.0
7	3.0	3.0	550	550	4.0
8	3.0	3.0	1000	1000	1.0

Northwestern University

A MULTIPLE-REGION THEORY OF INCOME AND TRADE[1]

By Lloyd A. Metzler

This paper deals with the effects of investment in one region or country upon income in all regions of an n-region system, and with the relations between these income movements and the pattern of trade among the various regions or countries. It includes both a static system of n equations based upon the usual definition of income and a corresponding dynamic system based upon the assumption that the output of a given region or country tends to rise when demand exceeds supply and to contract when supply exceeds demand. Under the assumed conditions, it is shown that stability of the system may be described in terms of Hicks's "conditions of perfect stability." The Hicks conditions, in turn, are dependent upon the marginal propensities to spend of the various regions. Throughout the discussion of the static problems, the system is assumed to be dynamically stable.

I. INTRODUCTION

The theory of employment and income that was developed during the decade of the thirties was concerned primarily with the economic forces governing the level of output in a closed economic system. From the outset, however, it was apparent that the new ideas had important applications to interregional and international problems. In particular, the theory of employment added considerably to our understanding of the mechanism by which an expansion or contraction of income in one region or country is transmitted to other regions or countries. But much of the early discussion of such problems was devoted to a highly simplified model in which the world was divided into two regions or countries; in this model an expansion or contraction of income was assumed to originate in one of the two regions or countries, and the repercussions upon income in the other region or country, and upon the balance of payments between the two, were then examined in some detail.[2] The purpose of the

[1] This paper was written in 1945 but was not submitted for publication because there seemed to be no widespread interest in the subject. In recent months, however, it has become apparent that the general principles of regional income movements are applicable to many other fields besides international trade. Most of the propositions developed in this paper, for example, are applicable to the theory of linear programming and to input-output studies within a single country. See, for instance: David Hawkins and Herbert A. Simon, "Note: Some Conditions of Macroeconomic Stability," Econometrica, Vol. 17, July–October, 1949, pp. 245–248; R. M. Goodwin, "The Multiplier as Matrix," *Economic Journal*, Vol. 59, December, 1949, pp. 537–555; and John S. Chipman, "The Multi-Sector Multiplier," appearing in this issue, pp. 355–374. In addition to these published papers, I have recently read an unpublished manuscript by H. A. John Green dealing with some aspects of the problem discussed in the present paper. In view of the renewed interest in the subject, it seems to me appropriate to present the results of my own investigation.

[2] See, for example, my own papers, "Underemployment Equilibrium in Inter-

present paper is to generalize the earlier discussion by considering a model of an economic system composed of n regions or countries, where n may be either large or small. Although I shall speak hereafter of "n countries," I assume it is clear that the conclusions apply without modification to the regions within a single country or, indeed, to any regional classification of the world economy, such as the economy composed of Eastern Europe, Western Europe, Latin America, and similar regions.

The procedure followed in this paper is essentially the same as that employed in the earlier discussions of the two-country model. The level of output in each of the n countries is assumed, initially, to be in a state of balance in the sense that the country's rate of output of goods and services is equal to the demand for such goods and services. A disturbance of the economic forces governing income is then assumed to take place in one of the countries, and the effects of this disturbance are traced throughout the n-country system. Both movements of real income or employment and movements of the international balance of trade are taken into account. In order to isolate the effects of employment and real income, the assumption is made that all prices, costs, and exchange rates remain unaltered. In other words, commodities and services are assumed to be produced and sold at constant supply prices. Exchange rates are assumed to be kept at fixed levels, either by central bank activity or by the normal operations of the gold standard. A free market for foreign exchange is postulated for each of the n countries, and imports are thus supposed to be limited by a country's income or purchasing power, and not by the size of its foreign-exchange reserves.

In the present world of unbalanced trade, dollar shortages, exchange controls, and "hard" or "soft" currencies, this last assumption will doubtless strike the reader as highly unrealistic. I should therefore add that the model of international trade discussed below is not intended as a description of the abnormal conditions prevailing today. Whether the model will or will not be a reasonable description of world trade and employment in the future is a question that can hardly be answered at the present time; the answer obviously depends upon numerous and unpredictable political influences as well as upon more narrow economic

national Trade," ECONOMETRICA, Vol. 10, April, 1942, pp. 97–112, and "The Transfer Problem Reconsidered," *Journal of Political Economy*, Vol. 50, June, 1942, pp. 397–414. See also F. Machlup, *International Trade and the National Income Multiplier*, Philadelphia: The Blakiston Co., 1943, 237 pp. Machlup presents an economic model involving three countries and has also described models involving a larger number. In his more complex models, however, a considerable amount of symmetry is assumed with respect to propensities to spend and to import, and for this reason his results cannot be regarded as completely general.

considerations, such as the fate of exchange controls, import quotas, and other governmental measures for controlling world trade. But whatever the future development of international trade may be, there are two reasons, it seems to me, why economic models such as the one given in this paper are useful. In the first place, there are almost certain to be large areas of the world, even in an economic system having extensive trade controls, in which payments between one region and another are made more or less freely. It is unlikely, for example, that any limitations other than the limitation of purchasing power will ever be placed upon transactions between Kansas and Nebraska or upon payments between the North Central States and the New England States in the United States. Likewise, payments between members of the sterling area of the British Commonwealth now occur quite freely despite the limitations upon payments outside the area.[3] Thus whatever happens to inter*national* trade, the model discussed below remains useful as a description of inter*regional* trade. The second and less important reason for regarding the model as useful is the fact that it can be helpful in interpreting economic events of the past. There have been long periods of time—the period under the gold standard before the first world war is an example—when international payments were made without restriction throughout the world. There is no doubt that during these periods limited income was the principal constraint upon imports, and the assumption made above regarding foreign exchange markets is accordingly appropriate for describing such periods.

The international theory of income to be presented below is, in at least two respects, a short-run theory. It is short-run, in the first place, in the same sense that Keynes's *General Theory* is a short-run theory: it takes the rate of current investment in each country either as a given amount or as a given function of income in that country, and makes no allowances either for the effects of continuous investment upon a country's capacity to produce or for the repercussions of a change in such capacity upon the demand for new investment. The theory, in brief, is a *static* theory of income and not a theory of growth; and for this reason it is obviously inapplicable over an extended period of economic development. The theory given below is short-run, in the second place, in its treatment of each country's balance of payments. The procedure followed in this regard is simply to investigate the effects of a given disturbance upon each country's balance of payments on current account, and not to

[3] This second example is perhaps slightly misleading, inasmuch as all of the countries concerned have far-reaching import controls which to a considerable extent take the place of exchange controls. It does not seem overly optimistic, however, to conjecture that the import controls within the area, if not those pertaining to imports from countries outside the area, will be gradually relaxed.

inquire about how a given deficit or surplus in this balance is offset. Nothing is said, in other words, about the role of capital movements in establishing and maintaining equilibrium in the flow of international payments and receipts. Thus, quite apart from the problems of growth, the position of equilibrium described below must be regarded as temporary. For, unless capital movements occur more or less automatically in response to discrepancies in a country's balance of payments on current account, a country with a deficit in its current account will sooner or later have to take measures such as cost deflation or currency depreciation to eliminate the deficit; and these measures, in turn, will affect the equilibrium of income. In other words, the equilibrium of income to be discussed in this paper can exist over a considerable period of time only if international monetary reserves are large or if capital movements are of the equilibrating type.

In demonstrating how an economic disturbance in one country affects income and employment throughout the world, any one of a considerable number of economic events could be selected as the disturbing force. We might, for example, investigate the repercussions of an increase in domestic investment in one of the n countries or of an increase in the consumption of domestic goods; or we might consider the effects of technological changes or changes in tastes which tend to shift the demand for goods and services in some particular country from domestic goods to imports; or we might, following more traditional lines, examine the economic consequences of reparations payments or some other form of income transfer between countries. The international repercussions of all such disturbances, however, have many common features, and it would, accordingly, be needlessly repetitious to consider each of them separately. Indeed, it seems to me that the important elements of an interregional or international theory of employment can, for the most part, be demonstrated by considering only one type of disturbance, namely, a change in domestic investment in one of the n countries. The effects of other, more complex, types of disturbances can then be determined by regarding these complex disturbances as combinations of movements of investment in one or more countries. Thus, for the purpose of income analysis, a reparations payment may be regarded as a combination of investment in the receiving country and disinvestment of the same amount in the paying country. In view of this possibility of transforming other disturbing forces into combinations of movements in investment, the international theory of employment presented below is developed entirely by considering the adjustment of the world economy to a change in investment in one country. The conclusions reached for this particular disturbance may readily be applied to other disturbances as well.

II. A SYSTEM OF INCOME EQUATIONS

Neglecting income transfers between countries, the current net income of a particular country is simply the market value of that country's net output of goods and services. The word "net" as used in this connection implies that two deductions are made from the total value of goods and services produced. First, the usual allowance is made for the depreciation of capital. Second, and more important for present purposes, the value of all imported goods and services employed in production is deducted from the market value of such production. This second deduction is necessary because a country's output incorporates not only the services of domestic factors of production, but also many materials and services purchased abroad; and the latter do not constitute income produced within the given country. The concept of income in an open economy is thus a sort of value-added-by-manufacture concept, except that the unit of account is a country or region rather than an industry.

Consider, now, a sum of values consisting of the following items: (1) all expenditures by the residents of a particular country upon consumers' goods and services, including imported as well as domestic goods and services; (2) net investment in plant, equipment, inventories, etc., including investment in equipment produced abroad as well as investment in things domestically produced; (3) exports of goods and services. In what respects does this sum differ from net income as defined in the preceding paragraph? The sum includes, in the first place, the value of imported materials and services employed in domestic production, and these obviously must be deducted in computing the net income produced within the given country. The sum also includes, in the second place, imported finished goods which may have been used either for consumption, for net investment, or for re-export; and since these imported finished goods obviously do not constitute a part of the particular country's current production, their value must likewise be deducted from the total in computing national income. Thus, we find that the total of domestic expenditures for consumption and investment plus receipts from exports exceeds national income by the value of imports, including both imports of finished goods and services and imports of intermediate goods and services. In terms of the final uses of goods and services, national income may accordingly be written as follows:

national income *equals* expenditures on consumers' goods and services
plus net investment *plus* exports of goods and services
less imports of goods and services.

Three of the items in this summation—consumption, net investment, and imports—are dependent upon the level of income and employment at home, while the remaining item, exports, depends upon income in all of

the countries to which the given country is selling goods and services.[4] This immediately suggests that for the world economy it might be convenient to set up a tabular presentation of income similar to the input-output tables developed by Leontief in the study of inter-industry relations.[5] Such a table would show how each country's income is *earned* —in sales at home and sales to other countries—and how income is *spent*—in purchases at home and purchases from other countries. Individual countries or regions, in other words, would replace the individual industries in the Leontief tables; and imports and exports would replace inputs and outputs.

Let $m_i(y_i)$ be the function which shows how *total* imports of the ith country from all other countries in the table are related to national income, y_i, of the importing country. This total import function will be composed of a number of subfunctions showing how imports from each of the other countries are related to income in the ith country. Thus, if $m_{ji}(y_i)$ represents the imports of the ith country from the jth country, stated as a function of income in the ith country, we will have $m_i(y_i) \equiv m_{1i}(y_i) + m_{2i}(y_i) + \cdots$, where the summation is extended over all countries from which the ith country imports goods or services. Since one country's imports are another country's exports, the entire pattern of world trade may be described in terms of the import functions, $m_{ji}(y_i)$. The tabular presentation of world income may then be completed by inserting functional relations for each country's expenditures on *all* goods and services. In setting up such total expenditure functions, there is no necessity to distinguish between consumers' goods and net investment, since the one affects income in the same way as the other. Suppose that both consumers' goods expenditures and net investment are dependent to some extent upon income at home, and let $u_i(y_i)$ represent such an expenditure function; $u_i(y_i)$, in other words, shows how expenditure in the ith country on *both* consumers' goods and net investment is related to the ith country's income. Hereafter, the function $u_i(y_i)$ will be called simply an "expenditure function"; it plays the same role in the present theory of employment that is usually attributed to the consumption

[4] If the import content of a country's exports differs from the import content of the goods and services produced for home use, total imports will depend not only upon income but also on the composition of income; i.e., upon the way output is divided between exports and goods or services produced for domestic use. Since the demand for the given country's exports is governed in part by income in other countries, and since imports in this instance depend partly upon exports, it follows that imports should really be expressed as a function of income in all countries. But this is a refinement which cannot be incorporated in the present model without complicating it unduly.

[5] W. W. Leontief, *The Structure of American Economy, 1919–1929,* Cambridge, Mass.: Harvard University Press, 1941, *passim.*

function. The quantity $u_i(y_i)$ represents *all* expenditures of the ith country on consumers' goods and net investment, irrespective of the source of goods and services purchased. It includes imported finished goods as well as the import-content of domestic production. In order to show how expenditure by a given country affects that country's net income, total imports, $m_i(y_i)$, must therefore be subtracted from the expenditure function, $u_i(y_i)$.

The foregoing relations are summarized in the accompanying table, which presents a hypothetical case of a world economy consisting of three countries. The items in a given *row* of this table provide a classification of the components of a country's national income according to the *sources* from which it was earned, while the items in the corresponding *column*

	Expenditures by Country 1 (1)	Expenditures by Country 2 (2)	Expenditures by Country 3 (3)	National Income (1) + (2) + (3)
Receipts from Sales by Country 1 (1)	$u_1(y_1) - m_1(y_1)$	$m_{12}(y_2)$	$m_{13}(y_3)$	y_1
Receipts from Sales by Country 2 (2)	$m_{21}(y_1)$	$u_2(y_2) - m_2(y_2)$	$m_{23}(y_3)$	y_2
Receipts from Sales by Country 3 (3)	$m_{31}(y_1)$	$m_{32}(y_2)$	$u_3(y_3) - m_3(y_3)$	y_3
Total Expenditures of Each Country (1) + (2) + (3)	$u_1(y_1)$	$u_2(y_2)$	$u_3(y_3)$	

indicate the *uses* of national income. The sum of the items in row 1 thus represents national income of Country 1, while the sum of the items in column 1 shows the total expenditures of Country 1 on all goods and services. In summing column 1, the positive items of imports, $m_{21}(y_1)$ and $m_{31}(y_1)$, will exactly cancel against total imports, which enter negatively in row 1, column 1, leaving only the total expenditure, $u_1(y_1)$.

Consider, now, a more general economic system consisting of n countries. Using the same notation as in the table, we can set up n equations which express the fact that, in equilibrium, each country's output is equal to the demand for this output. Thus we have

$$
(1)\quad
\begin{aligned}
y_1 &= [u_1(y_1) - m_1(y_1)] + m_{12}(y_2) + \cdots + m_{1n}(y_n), \\
y_2 &= m_{21}(y_1) + [u_2(y_2) - m_2(y_2)] + \cdots + m_{2n}(y_n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
y_n &= m_{n1}(y_1) + m_{n2}(y_2) + \cdots \\
&\qquad + [u_n(y_n) - m_n(y_n)].
\end{aligned}
$$

Since there are n countries in all, these n equations are sufficient, with given prices and exchange rates, to determine the level of income in each country.

III. STABILITY OF THE SYSTEM

Equations (1) are *static* equations; they indicate the levels of income which the system would achieve if consuming and investing habits remained unchanged over a sufficient period of time. They are accordingly useful in solving economic problems such as the ones mentioned earlier. Suppose, for example, that the propensity to consume or to invest in domestic goods were to increase in Country 1; the demand for goods and services would then rise throughout the world economy, and if the system were stable a new equilibrium, corresponding to the higher level of demand in Country 1, would eventually be established in all countries. Equations (1) enable us to show how the new position of equilibrium in each of the n countries compares with the old. This problem is nothing more than a generalization of the familiar investment multiplier.

Before discussing the static theory of income, however, there is a closely related dynamic problem that it will be useful to discuss first. As we shall see, solution of the dynamic problem provides a considerable amount of information about the static theory. If income in one or more of the n countries is not in a state of equilibrium, so that the current level of output differs from the current demand, then the current levels of income, $y_1, y_2, \cdots, y_n$, will not satisfy equations (1). In some instances, the level of output or income, y_i, may fall short of demand, while in other instances output will probably be in excess of current demand. Under these circumstances there will be a tendency for the level of output in each country to change, as producers try to bring their production plans in line with current requirements. The changes in output, in turn, will alter the level of income in each country, thereby bringing about shifts of demand and creating further discrepancies between supply and demand. The dynamic problem I wish to discuss is whether such a system has a natural tendency to approach a balanced state or whether discrepancies between demand and supply tend to produce still larger discrepancies. In short, is the system of income equations stable or unstable?

In order to answer this question, some assumptions must be made as to what happens in each country when output differs from current demand.

Although no simple model can possibly do justice to such a complex problem, it seems to me reasonable to suppose that producers as a group will react to a discrepancy between output and demand by altering the rate of output. I shall therefore assume that output, and hence income, increases whenever demand exceeds current output and falls when demand is less than current output. Moreover, I shall also assume that the speed with which output plans are altered is directly proportional to the size of the discrepancy between demand and supply; a big discrepancy, in other words, leads to a more rapid response than a small one. Although this second assumption is not absolutely essential, it is an assumption which will simplify our problem somewhat without altering the results in any important respects. Throughout the period of time when income is out of equilibrium, discrepancies between demand and supply are assumed to be met by appropriate adjustments of business inventories.

For any given country, say Country 1, the rate of current net output or national income is y_1, while the net demand for this output is $u_1(y_1) - m_1(y_1) + m_{12}(y_2) + \cdots + m_{1n}(y_n)$. The preceding assumptions concerning the behavior of producers may therefore be embodied, as a first approximation, in the following system of dynamic equations:

$$\begin{aligned}
\frac{dy_1}{dt} &= k_1[u_1(y_1) - m_1(y_1) + m_{12}(y_2) + \cdots + m_{1n}(y_n) - y_1],\\
\frac{dy_2}{dt} &= k_2[u_2(y_2) - m_2(y_2) + m_{21}(y_1) + \cdots + m_{2n}(y_n) - y_2],\\
&\;\;\vdots\\
\frac{dy_n}{dt} &= k_n[u_n(y_n) - m_n(y_n) + m_{n1}(y_1) + \cdots\\
&\qquad + m_{n,n-1}(y_{n-1}) - y_n].
\end{aligned} \tag{2}$$

The constants, k_i, in these equations are positive numbers which represent the speeds of adjustment of output in the various countries.

Equations (2) cannot be solved without knowing the explicit form of the expenditure functions and import functions. Since we are primarily interested in the stability of the system and not in its explicit solution, however, we may consider only a linear approximation to (2). Stability of the linear approximation is obviously a necessary condition, although not always a sufficient condition, for stability of equations (2). Expanding the right-hand side of (2) in a Taylor expansion about the equilibrium values $y_1^0, y_2^0, \cdots, y_n^0$, and dropping all except linear terms, we have

$$\begin{aligned}
\frac{dy_1}{dt} &= k_1(u_1' - m_1' - 1)(y_1 - y_1^0)\\
&\qquad + k_1 m_{12}'(y_2 - y_2^0) + \cdots + k_1 m_{1n}'(y_n - y_n^0),
\end{aligned}$$

[Equations (3) continued on p. 338]

$$
\begin{aligned}
(3)\quad \frac{dy_2}{dt} &= k_2 m'_{21}(y_1 - y_1^0) \\
&\quad + k_2(u'_2 - m'_2 - 1)(y_2 - y_2^0) + \cdots + k_2 m'_{2n}(y_n - y_n^0), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{dy_n}{dt} &= k_n m'_{n1}(y_1 - y_1^0) \\
&\quad + k_n m'_{n2}(y_2 - y_2^0) + \cdots + k_n(u'_n - m'_n - 1)(y_n - y_n^0),
\end{aligned}
$$

where $u'_i \equiv (du_i/dy_i)_{y_i^0}$, $m'_{ji} \equiv (dm_{ji}/dy_i)_{y_i^0}$, etc. Equations (3), being linear with constant coefficients, can be solved for any given initial conditions so as to express each of the incomes, y_i, as a function of time, as follows:

$$
(4)\qquad y_i(t) = y_i^0 + A_{i1}e^{\lambda_1 t} + A_{i2}e^{\lambda_2 t} + \cdots + A_{in}e^{\lambda_n t},
$$

where the A_{ij} are constants dependent upon the initial value of income at time $t = 0$, and where the λ_j are roots of the following equation:

$$
(5)\qquad \begin{vmatrix} k_1(1 + m'_1 - u'_1) + \lambda & -k_1 m'_{12} & \cdots & -k_1 m'_{1n} \\ -k_2 m'_{21} & k_2(1 + m'_2 - u'_2) + \lambda & \cdots & -k_2 m'_{2n} \\ \vdots & \vdots & & \vdots \\ -k_n m'_{n1} & -k_n m'_{n2} & \cdots & k_n(1 + m'_n - u'_n) + \lambda \end{vmatrix} = 0.
$$

In order for $y_i(t)$ to approach its equilibrium value, y_i^0, as t increases, it is apparent from (4) that the real parts of $\lambda_1, \lambda_2, \cdots, \lambda_n$ must all be negative. The necessary and sufficient conditions for this to be true may conveniently be expressed in terms of the following nth-order determinant:

$$
(6)\qquad M = \begin{vmatrix} 1 + m'_1 - u'_1 & -m'_{12} & \cdots & -m'_{1n} \\ -m'_{21} & 1 + m'_2 - u'_2 & \cdots & -m'_{2n} \\ \vdots & \vdots & & \vdots \\ -m'_{n1} & -m'_{n2} & \cdots & 1 + m'_n - u'_n \end{vmatrix}
$$

The coefficient m'_{ij} of the determinant (6) represents, of course, the marginal propensity of the jth country to import from the ith country; i.e., it shows how the demand in Country j for imports from Country i is affected by a small increase in the former country's income. Similarly, the coefficient m'_j represents the marginal propensity of the jth country to import from all other countries together, so that $m'_j \equiv m'_{1j} + m'_{2j} + \cdots + m'_{nj}$. Throughout this paper, coefficients such as m'_{ij} are assumed to be positive or zero, which means that all of the off-diagonal elements of M are negative or zero.[6] The coefficient, u'_j, represents the marginal

[6] If one country's imports from another consisted predominantly of inferior commodities, the former's propensity to import from the latter might conceivably

propensity of the jth country to spend, including the marginal propensity to invest, if any, as well as the marginal propensity to consume, and including expenditure on imported finished goods as well as upon domestic goods. Normally u_j' will be less than unity, but, if the propensity to invest is large, this need not be true.

I have demonstrated in an earlier paper that, for dynamic systems such as (3) in which all off-diagonal coefficients of the y_i are positive or zero, the necessary and sufficient conditions of stability are identical with the so-called Hicksian conditions of perfect stability.[7] This means that the determinant, M, and any set of its principal minors such as

$$1 + m_i' - u_i', \qquad \begin{vmatrix} 1 + m_i' - u_i' & -m_{ij}' \\ -m_{ji}' & 1 + m_j' - u_j' \end{vmatrix},$$

$$\begin{vmatrix} 1 + m_i' - u_i' & -m_{ij}' & -m_{ik}' \\ -m_{ji}' & 1 + m_j' - u_j' & -m_{jk}' \\ -m_{ki}' & -m_{kj}' & 1 + m_k' - u_k' \end{vmatrix},$$

etc., must be positive. Hereafter, any determinant satisfying these conditions will be called a "Hicksian determinant."[8]

Since the speeds of adaptation, k_j, do not appear in the Hicks conditions, it follows that stability of (3) is independent of such speeds. A system which is stable for one set of speeds of adaptation will therefore be stable for all other possible sets. The fact that producers in one country change their production plans more rapidly than producers in another country has no effect upon the stability of the system.

Having established a general set of conditions which must be fulfilled in order that the income equations shall be stable, it is possible to go a step further and show that these Hicksian conditions depend, in a unique

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be negative. In this event many of the theorems of the present paper would be invalid. The presence of negative propensities to import makes the conditions of stability considerably more complicated. Compare, for example, my conclusions concerning stability with those of John S. Chipman in the paper which follows this one.

[7] L. A. Metzler, "Stability of Multiple Markets: The Hicks Conditions," ECONOMETRICA, Vol. 13, October, 1945, pp. 277–292.

[8] In my earlier paper the conditions of stability were expressed in terms of a determinant whose elements all had signs opposite to the signs of the corresponding elements of M. As a result, the formal appearance of the stability conditions was not the same as in the present paper. In the terminology of my earlier paper, stability of the system required that the principal minors, when arranged as above, should be alternately negative and positive, and that the basic determinant itself should have the sign of $(-1)^n$. By changing the sign of each of the elements of M, the reader can easily verify that these earlier stability conditions are identical with the ones given in the present paper.

way, upon the propensities to spend in all countries. In particular, two propositions will be demonstrated. First, if the marginal propensity to spend, including expenditure on investment goods as well as on consumers' goods, is less than unity in every country, the system is necessarily Hicksian and therefore stable. Second, if the marginal propensity to spend is *greater* than unity in every country, the system cannot be Hicksian and must therefore be unstable.

To prove these propositions, it is convenient to use a theorem developed by Mosak.[9] Mosak's theorem, in slightly modified form, is as follows: If an nth-order determinant is Hicksian, and if the off-diagonal elements $-m'_{ij}$ are all negative, then the cofactor, M_{ij}, of the element $-m'_{ij}$ is positive for all i and j. The proof of this theorem is a simple proof by induction. Expanding M_{ij} about the row containing the elements $-m'_{j1}, -m'_{j2}, \cdots, -m'_{jn}$, we may write

$$M_{ij} \equiv \sum_k - m'_{jk} M_{ij,jk}, \tag{7}$$

where $M_{ij,jk}$ is the cofactor of the element $-m'_{jk}$ in the determinant, M_{ij}, and where the summation extends over all values of k from 1 to n except $k = j$. Since $M_{ij,jk} \equiv -M_{jj,ik}$, (7) may be written as follows:

$$M_{ij} \equiv \sum_k m'_{jk} M_{jj,ik}. \tag{8}$$

Now M_{jj} is a Hicksian determinant of order $n - 1$. Suppose that Mosak's theorem is true for such an $(n - 1)$th-order determinant. Then $M_{jj,ik}$ is positive, and it follows, from (8), that M_{ij} must likewise be positive. Thus, if the theorem is true for the cofactors of an $(n - 1)$th-order determinant obtained by deleting the jth row and jth column of M, it is also true for the cofactors of the nth-order determinant, M. A similar argument applies, of course, to the cofactors of any lower-order Hicksian determinants obtained from M by deleting like rows and columns. To complete the proof we must show that the theorem is true for a low-order principal minor of M, such as a second-order minor. A typical second-order minor of M is

$$\begin{vmatrix} 1 + m'_i - u'_i & -m'_{ij} \\ -m'_{ji} & 1 + m'_j - u'_j \end{vmatrix}$$

The cofactors of the off-diagonal elements of this minor are m'_{ij} and m'_{ji}, respectively, and these are both positive. Thus, Mosak's theorem is proved; i.e., we have shown that if the nth-order determinant is Hicksian, the cofactors of its off-diagonal elements are all positive.

[9] Jacob L. Mosak, *General-Equilibrium Theory in International Trade*, Cowles Commission Monograph No. 7, Bloomington, Ind.: The Principia Press, 1944, pp. 49–51.

With the aid of this theorem, the two propositions stated above concerning the relations between marginal propensities to spend and the determinant, M, may easily be proved. Consider, first, the case in which the marginal propensity to spend is *less* than unity in each country. According to our first proposition, the determinant M is necessarily Hicksian and the dynamic system (3) is therefore stable under these conditions. The proposition will be proved by induction. Since $m_i' \equiv m_{1i}' + m_{2i}' + \cdots + m_{ni}'$, it is clear that the sum of the elements of the ith column of M is equal to $1 - u_i'$, where u_i' is the marginal propensity to spend of the ith country. Thus, if all u_i' are less than unity, the sum of the elements of each column of M will be positive. Adding all other rows of M to the first row, we may write:

$$(9) \qquad M \equiv \left| \begin{array}{c|c} 1 - u_1' & 1 - u_2' \quad 1 - u_3' \cdots 1 - u_n' \\ \hline -m_{21}' & \\ -m_{31}' & M_{11} \\ \cdots & \\ -m_{n1}' & \end{array} \right|$$

where M_{11} denotes the cofactor of M obtained by deleting the first row and first column. Now it is evident that under our assumed conditions M_{11} is an $(n - 1)$th-order determinant having the same essential characteristics as M itself; i.e., the sum of the elements of each column of M_{11} is positive. The first column of M_{11}, for example, contains all of the elements of the corresponding column of M except the negative quantity, $-m_{12}'$, and similarly for all other columns. It follows that if the sum of the elements of a given column of M is positive, the same will be true a fortiori of the sum of the elements in the corresponding column of M_{11}. Any theorems concerning M which are based upon this characteristic will therefore be equally applicable to M_{11}. And a similar argument applies to lower-order principal minors of M, such as $M_{11,22}$, $M_{11,22,33}$, etc.

Suppose, now, that our theorem is true for the $(n - 1)$th-order determinant, M_{11}; i.e., suppose that M_{11} is Hicksian. It can then be shown that the nth-order determinant, M, is also Hicksian. Expanding (9) on the first row and first column, in a Cauchy expansion, we find:[10]

$$(10) \qquad M \equiv (1 - u_1')M_{11} + \sum_k \sum_j m_{j1}'(1 - u_k')M_{11,jk}.$$

If M_{11} is a Hicksian determinant it must be positive, and $M_{11,jj}$, $M_{11,kk}$, etc., must likewise be positive. Moreover, by Mosak's theorem, $M_{11,jk}$ is positive. Since the m_{j1}' are positive or zero, and since $1 - u_1'$ and $1 - u_k'$

[10] See A. C. Aitken, *Determinants and Matrices*, New York: Interscience Publishers, Inc., 1944, pp. 74–75.

are positive by hypothesis, it follows immediately from (10) that, if M_{11} is a Hicksian determinant, M is positive and is therefore Hicksian.

It has now been demonstrated that if all u'_k are less than unity, and if M_{11} is Hicksian, then M is likewise Hicksian. By a similar argument it can be shown that, if $M_{11,22}$ is Hicksian and if the u'_k are less than unity, M_{11} is necessarily Hicksian. To complete the proof that M is always a Hicksian determinant when the marginal propensity to spend, u'_k, is less than unity in every country, it is sufficient to show that the theorem is true for any low-order principal minor of M. Consider, for example, the following second-order minor:

$$\begin{vmatrix} 1 + m'_i - u'_i & -m'_{ij} \\ -m'_{ji} & 1 + m'_j - u'_j \end{vmatrix}.$$

Since $m'_i \geqslant m'_{ji}$ and $m'_j \geqslant m'_{ij}$, it is easy to show by expanding the above determinant that it is necessarily positive whenever u'_i and u'_j are both less than unity. Moreover, it may be seen by inspection that, under the prescribed conditions, the principal minors are positive. The second-order minor of M is therefore Hicksian, and our proof that M is a Hicksian determinant is complete.

If M is a Hicksian determinant, it follows from the results of my earlier paper that the dynamic system represented by equations (3) is a stable system. This conclusion will perhaps not surprise anyone, since it is simply a generalization of the theory of income stability of a single, closed economic system. It is well known that the multiplier in such a one-country system cannot have a finite value unless the country's marginal propensity to spend is less than unity. I have now established an analagous condition—sufficient but not necessary—for the case of an n-country economy.

Consider, now, an extreme case in which the marginal propensity to spend is *greater* than unity in every country. I have suggested above that in this event the determinant M cannot be Hicksian and the dynamic system (3) must therefore be unstable. The proof of this proposition consists of showing that if all u'_k exceed unity the assumption that M is Hicksian involves a contradiction. If M is Hicksian, the principal minor M_{11} is, of course, also Hicksian, which means that $M_{11,jk}$ and M_{11} are both positive. But if the marginal propensity to spend is greater than unity in all countries, $1 - u'_k$ is negative for all values of k. From (10) it follows that M must be negative. This contradicts the assumption that M is a Hicksian determinant and proves, in fact, that M cannot be Hicksian. It shows, in other words, that if the determinant is Hicksian so far as its principal minors are concerned, and if all marginal propensities to spend exceed unity, the determinant itself is negative and is therefore non-Hicksian. Employing again the results of my previous paper, it is

clear that under such conditions the dynamic system (3) must necessarily be unstable.

I have now examined the stability of income for two different situations. The first, which might be called the normal situation, is the case in which the marginal propensity to spend is less than unity in every country. The second, which goes to the opposite extreme, is the case in which every country has a marginal propensity to spend exceeding unity. In the first situation the system was found to be Hicksian, and therefore stable, while in the second it was found to be non-Hicksian and therefore unstable. Between these two extremes may be found a large number of intermediate situations in which the propensity to spend is less than unity in some countries and greater than unity in others. The basic determinant, M, of these intermediate systems may or may not be Hicksian, which means that the systems may or may not be dynamically stable. Broadly speaking, we may say that M will be Hicksian and the system will be stable if the countries with low propensities to spend dominate, while in the converse case M will be non-Hicksian and the system unstable. In any event, the discussion that follows in Sections IV and V below concerning the international repercussions of added investment in one of the n countries is based upon the explicit assumption that the income equations form a dynamically stable system. In other words, the assumption is made that an increase of investment in one of the countries leads ultimately to a new equilibrium of income in all countries, and does not set off a continuous process of expansion culminating in a runaway inflation. This means that, while the propensity to spend may exceed unity in some countries, it cannot do so in all countries; at least one of the countries must have a propensity to spend of less than unity, and the low-propensity countries must be sufficiently important so that the basic determinant, M, is a Hicksian determinant.

IV. INVESTMENT AND INCOME

Having examined the conditions of stability of our income equations, we are now in a position to investigate some problems of comparative statics. Suppose that national income is initially in equilibrium in all countries and that this equilibrium is disturbed by an increase of investment in one of the countries, say in Country 1. If the increase of investment is sustained over a sufficient period of time, and if the income equations are dynamically stable, a new equilibrium corresponding to the higher rate of investment will eventually be established throughout the system. The income of every country will probably be affected to some extent by the expansion of investment in Country 1; and, as national incomes are altered, each country's exports and imports, or its balance of payments on current account, will likewise be changed. The

present section is concerned with the changes in income brought about by the higher level of investment in Country 1.

Let α_1 represent autonomous or noninduced investment in Country 1. The first equation of the static system (1), including the additional investment, then becomes:

$$(11) \qquad y_1 = u_1(y_1) - m_1(y_1) + m_{12}(y_2) + \cdots + m_{1n}(y_n) + \alpha_1 .$$

Assuming no change in autonomous investment in the other countries, the remaining $n - 1$ equations of (1) are unaltered. Equation (11) and the last $n - 1$ equations of (1) thus form a closed system of n equations in which the income of each country may be regarded as a function of α_1. In order to see how the increase of investment in Country 1 affects each country, we may differentiate (11) and the last $n - 1$ equations of (1) with respect to α_1, and solve the resulting linear equations for $dy_1/d\alpha_1$ and $dy_k/d\alpha_1$. It will then be found that

$$(12) \qquad \frac{dy_1}{d\alpha_1} = \frac{M_{11}}{M}, \qquad \frac{dy_k}{d\alpha_1} = \frac{M_{1k}}{M},$$

where, as before, M is the determinant of marginal propensities given by (6). Now, we know from the conditions of stability and from Mosak's theorem that M_{11}, M_{1k}, and M must all be positive. Both $dy_1/d\alpha_1$ and $dy_k/d\alpha_1$ must therefore be positive, which shows that an increase in investment in one of the n countries increases the level of income in every country in the system. There is, of course, nothing startling or profound about this conclusion; indeed, it is a conclusion which could have been reached intuitively without any mathematics at all.[11] It is therefore important only insofar as it leads to less obvious relations.

The expression, M_{11}/M, which shows how income in the first country is affected by an increase of investment in that country, is a generalized form of investment multiplier. I wish to show, now, how this generalized multiplier is related to two simpler multipliers that one encounters frequently in the theory of employment. The first of these simple multipliers is the ordinary investment multiplier of a closed economic system, i.e., the multiplier which ignores foreign-trade leakages; the second is the

[11] Any economist who gives the matter any thought will probably feel that to develop the rather complicated theorems of Section III concerning Hicksian determinants and conditions of stability simply in order to prove that an increase in investment in one country causes income to rise in all countries is like using a bulldozer to move an ant hill. His intuitive feeling may be so strong, in fact, that he will prefer to reverse the procedure of the present paper and use what he "knows" about the economic system to prove the theorems concerning determinants in Section III! While the mathematician will doubtless object to this procedure as completely lacking in rigor, I must confess that I have considerable confidence in it, particularly since it was substantially such a trend of thought which first led me to suspect the truth of the mathematical propositions of Section III above.

so-called foreign trade multiplier, which makes allowance for foreign-trade leakages but does not take into account the effects of income movements in other countries upon the demand for a given country's exports. If, as before, u_1' denotes the marginal propensity to spend of the first country, and m_1' denotes that country's marginal propensity to import, the ordinary investment multiplier, which assumes that all demand is for home goods, is simply $1/(1 - u_1')$. The foreign trade multiplier, on the other hand, is $1/(1 - u_1' + m_1')$. What is the relation of these two simple multipliers to the generalized multiplier given by (12)? Using the stability conditions and Mosak's theorem, it may be shown that, in the normal case in which the marginal propensity to spend is less than unity in every country, the value of the generalized multiplier lies between the ordinary multiplier and the foreign trade multiplier. To prove this proposition, notice first that by adding all other rows to the first row of M, expanding on the elements of this new row, and dividing both numerator and denominator by M_{11}, we may write:

$$(13)\quad \begin{aligned} \frac{dy_1}{d\alpha_1} &= \frac{M_{11}}{M} \\ &= \frac{1}{(1 - u_1') + [(1 - u_2')M_{12}/M_{11}] + \cdots + [(1 - u_n')M_{1n}/M_{11}]}. \end{aligned}$$

Since M_{1k}/M_{11} is positive for any value of k, and since all of the u_k' are assumed to be less than unity, it is clear that the expression in (13) is less than the ordinary investment multiplier, which in this instance has a value of $1/(1 - u_1')$.

The second limit to $dy_1/d\alpha_1$ may be found by expanding M on its first *column* and again dividing both numerator and denominator of the resulting expression for $dy_1/d\alpha_1$ by M_{11}. It will then be found that

$$(14)\quad \frac{dy_1}{d\alpha_1} = \frac{1}{1 - u_1' + m_1' - [m_{21}'M_{21}/M_{11}] - \cdots - [m_{n1}'M_{n1}/M_{11}]}.$$

Again, since M_{k1}/M_{11} is positive, the value of $dy_1/d\alpha_1$ given by (14) is clearly *greater* than the foreign trade multiplier, $1/(1 - u_1' + m_1')$. Thus, I have shown that in the normal case in which all marginal propensities to spend are less than unity, the generalized investment multiplier has the following limits:

$$(15)\quad \frac{1}{1 - u_1' + m_1'} < \frac{dy_1}{d\alpha_1} < \frac{1}{1 - u_1'}.$$

These limits derive their importance from the fact that they represent two forms of the multiplier which have played prominent roles in the historical development of the theory of employment.

If one or more of the other countries—i.e., Countries 2, 3, $\cdots$, n—has

a marginal propensity to spend greater than unity, one of the limits given by (15) *may* not hold. In particular, while the generalized multiplier is always greater than the foreign trade multiplier, as (14) shows, it may in special cases also be greater than the ordinary investment multiplier. Consider, for example, the following system:

$$(16) \qquad y_1 = 0.4y_1 + 0.5y_2 + \alpha_1\,, \qquad y_2 = 0.2y_1 + 0.7y_2\,.$$

For this system, $dy_1/d\alpha_1 = 3.75$, while $1/(1 - u_1') = 2.5$. Thus, when the marginal propensity to spend of one or more of the "other" countries exceeds unity, the true investment multiplier for a given country may be larger than the ordinary investment multiplier. In most cases, however, it seems probable that the true multiplier will lie between the two simple multipliers, as indicated in (15).

It may be useful at this point to give a brief intuitive explanation of the relations between the three multipliers. The foreign trade multiplier is the smallest of the three because it assumes that a country's exports are given and independent of its imports. In a period of rising domestic income, in other words, the foreign trade multiplier tacitly assumes that increased expenditures on imports represent net leakages from the country's circular flow of income; no allowance is made for the fact that as imports rise the level of income in other countries also rises, and the demand for the particular country's exports therefore rises, to some extent, along with its imports. The generalized multiplier takes account of this secondary rise in the country's exports, and it is therefore larger than the foreign trade multiplier. The ordinary investment multiplier, on the other hand, makes no allowance either for the leakages from the circular flow of income arising from increased imports or for the return of some of these leakages in the form of increased exports; it assumes, instead, that every increase in expenditure represents an equivalent increase in domestic income. Now, since the secondary rise in exports is normally smaller than the increase in imports with which it is associated, it follows that foreign trade usually exerts a retarding effect upon a rise in income originating in domestic investment. In short, the effect of foreign trade is to spread the stimulating effects of investment in one country over the entire economic system, thereby diluting to some extent the stimulus to income in the country originating the expansion. And, because it ignores this diluting effect, the ordinary investment multiplier overstates the rise in income at home to be expected from a given increase in domestic investment.

V. INVESTMENT AND THE PATTERN OF TRADE

So much for the effects of investment upon income and employment. I turn now to the related problem of the pattern of trade. As income

expands throughout the system, each country's exports and imports will rise, and it is almost inevitable under such conditions that the balance of trade of most if not all of the countries will be affected. In the new position of equilibrium, some countries will have more favorable balances while others will have less favorable balances than in the old. What can be said, in a general way, about the new network of trade compared with the old?

With respect to bilateral balances between individual pairs of countries, there is very little that a general theory such as the one outlined in this paper can predict. The outcome depends entirely upon the particular values of the propensities to spend and to import, and may show wide variation from one economic system to another. With respect to each country's balance of trade as a whole, on the other hand, certain broad generalizations are possible. In particular, we can specify the conditions under which a general expansion originating in Country 1 is likely to lead to an improvement or to a deterioration in a given country's balance of trade with the rest of the world. Since there is no difficulty in forecasting how a given expansion will *initially* affect the balance of international payments, the problem before us is essentially a problem of comparing the initial, or primary, effects with the secondary repercussions. We want to know, in particular, whether the secondary repercussions are likely to reinforce or to offset the primary effects. Consider, for example, the balance of payments of some country other than Country 1, say Country k. As investment and income expand in Country 1, the initial effect will probably be an increase in exports from Country k to the expanding country, thereby giving the latter a temporary surplus in its balance of payments. A similar initial effect may be anticipated, of course, in all of the other countries dealing with Country 1. But, as the other countries' exports to Country 1 rise, their incomes will also rise, and the increase in incomes, in turn, will increase the demand for imports in these countries. The secondary income movements thus tend to offset the initial changes in balances of payments of the other countries.[12] There is no obvious reason, however, why the offsetting movement in each country's balance of payments should always be exactly equal to the initial disturbance. In the new equilibrium some countries will probably have more favorable balances of payments while others will have less favorable ones. What are the circumstances that distinguish the "surplus" countries from the "deficit" countries?

[12] It was no doubt this offsetting tendency that Nurske had in mind when he said that the theory of employment provides both an explanation of the adjusting process of the balance of payments and a theory of the transmission of business cycles from one country to another. (Ragner Nurske, "Domestic and International Equilibrium," in *The New Economics*, S. E. Harris, ed., New York: Alfred A. Knopf, Inc., 1947, p. 264.)

The question may be answered by considering the interrelations between balances of payments and incomes. Although the balance of trade of a given country depends upon the incomes of all countries in the system, there is a convenient way of relating each country's balance of trade to the *income of that country alone*. Thus, from the definition of national income given in (1) above, it follows that the excess of a country's exports over its imports is equal to the excess of its national income over its total expenditure on both consumers' goods and net investment. This is no more than a technical way of stating the common-sense proposition that a country with an export surplus is producing more than it uses itself, while a country with an import surplus is using more than it produces. But it is a technique, as we shall see, which saves a good deal of tedious algebra. Consider, for example, the balance of payments of Country k. If b_k denotes this balance, then it is clear from (1) that

$$b_k = y_k - u_k(y_k), \tag{17}$$

whence

$$\frac{db_k}{d\alpha_1} = (1 - u_k') \frac{dy_k}{d\alpha_1}. \tag{18}$$

Since $dy_k/d\alpha_1$ is positive, (18) shows that the direction of change of Country k's balance of payments depends upon that country's marginal propensity to spend. If its propensity to spend is less than unity, as will normally be the case, the balance of payments of Country k will be improved by the expansion in Country 1 even after allowing for the secondary rise of imports. But if the country's propensity to spend is *greater* than unity, (18) shows that its balance of payments on current account will be worsened by the expansion in Country 1. In this instance, the secondary rise of Country k's imports will be *more* than sufficient to offset the initial rise of its exports.

Now suppose that the marginal propensity to spend of each of the countries $2, 3, \cdots, n$ is less than unity. Under such conditions, the expansion of income in Country 1 improves the trade balances of all other countries in the system; and from this it follows that the trade balance of the country initiating the expansion must be less favorable than before the expansion began. In short, an expansion of income originating in one country normally moves the balance of trade *against* that country and *in favor* of all other countries in the system; as long as marginal propensities to spend are all less than unity, this proposition holds true regardless of the relative sizes of the marginal propensities to import. For this reason we cannot say that, if the other countries' propensities to import from Country 1 are high, the induced expansion of their imports is likely to over-balance the initial rise of their exports,

leaving them with less favorable trade balances than before the expansion began. The outcome depends not upon the relative magnitudes of import propensities, but upon the absolute size of each of the propensities to spend. If the marginal propensities to spend are less than unity, the result will be an improvement in the balances of payments of all countries except Country 1, irrespective of the size of import propensities.

If marginal propensities to spend in some of the countries exceed unity, on the other hand, it is possible that some or all of the conclusions of the preceding paragraph will have to be reversed. Consider first an extreme case. Suppose that the propensities to spend exceed unity in *all* of the countries 2, 3, $\cdots$, n. Under these circumstances it is clear from (18) that the balance of trade of each of these countries would become less favorable as a result of expansion in Country 1; the secondary rise of imports in each of the countries would overbalance the primary increase in exports. But if Countries 2, 3, $\cdots$, n all have less favorable balances of payments, Country 1 must necessarily have a *more* favorable balance. After allowing for all repercussions, in other words, expansion of income in Country 1 increases that country's exports more than its imports are increased. Public works, encouragement of private investment, and other measures to expand the employment of resources in Country 1 would not, under the circumstances, create a balance-of-payments problem for the expanding country. Each time Country 1 increased its imports it could count upon an even larger secondary increase in its exports.

It is conceivable that this conclusion would be valid even under less extreme circumstances. Suppose, for example, that some of the countries 2, 3, $\cdots$, n had propensities to spend greater than unity while others had spending propensities less than unity. From (18) it is clear that some of these countries would then suffer a worsening of their balances of payments when Country 1 started an expansion, while others would find their balances of payments improved. And if the sum of all the adverse and favorable changes together were adverse, then Country 1 would obviously have a more favorable balance of payments than in the initial equilibrium. On the other hand, if the sum of changes in the balances of payments of Countries 2, 3, $\cdots$, n were favorable, then the movement of Country 1's balance would necessarily be adverse. Thus, when some of the spending propensities of Countries 2, 3, $\cdots$, n exceed unity, while others are less than unity, it is impossible without additional information to predict the effect of expansion on the balance of payments of the country initiating the expansion. The outcome depends upon a balancing of forces, i.e., upon a balancing of the influence of the stable countries against the influence of the unstable ones.

Thus far we have regarded the balance of payments of Country 1 as a sort of residual; we have described its movement only after seeing what

happened to the balances of payments of the other countries in the system. Although this procedure is satisfactory for some purposes, it does not allow us to say much about the *magnitude* of the movement in Country 1's balance of payments. It is therefore useful to examine this balance directly. From (11) and (1) the balance of payments of Country 1 may be written as follows:

$$b_1 = y_1 - u_1(y_1) - \alpha_1 . \tag{19}$$

In words, this says that Country 1's balance of payments on current account is the difference between its income and its total expenditure on goods and services, *including in the latter autonomous expenditures*, α_1, as well as $u_1(y_1)$. Differentiating b_1 with respect to α_1, we find:

$$\frac{db_1}{d\alpha_1} = (1 - u_1') \frac{dy_1}{d\alpha_1} - 1. \tag{20}$$

In evaluating (20) we may begin with what I have called the normal case, namely, the case in which all marginal propensities to spend are less than unity. In this case we know from Section IV above that $dy_1/d\alpha_1$ is less than the ordinary investment multiplier; i.e., it is less than $1/(1 - u_1')$. From this fact we can derive the following limits for the movement of the balance of payments on current account of Country 1:

$$-1 < \frac{db_1}{d\alpha_1} < 0. \tag{21}$$

The limits given by (21) show that in the normal case an increase of investment in Country 1 moves the balance of payments on current account *against* the expanding country; and the amount of the unfavorable movement is normally less than the increase of investment. A one billion dollar public works program consisting exclusively of expenditure on domestic goods and services, for example, could not under normal circumstances create a foreign-trade deficit in the expanding country greater than the amount of public works.

If the marginal propensity to spend in the expanding country were *greater* than unity, however, the limits given by (21) would no longer apply. It is apparent from (20) that under such a condition $db_1/d\alpha_1$ would be less, algebraically, than -1. The unfavorable movement of Country 1's balance of payment on current account would thus be *greater* than the amount of autonomous investment. An economy characterized by such a high propensity to spend would, of course, be highly unstable, and its instability, in turn, would lead to frequent and severe balance-of-payments problems vis-a-vis the rest of the world.

If the instability is in the rest of the world rather than in Country 1, there may be no balance-of-payments problem at all in the country initi-

ating the expansion. In other words, if a larger number of the "other countries" have marginal propensities to spend greater than unity, while Country 1 has a propensity to spend *less* than unity, (20) shows that the change in the balance of payments of Country 1 may be favorable rather than unfavorable. This would be true whenever $dy_1/d\alpha_1$ were greater than $1/(1 - u_1')$. In such a situation the secondary rise in exports of the expanding country would exceed the rise in imports; the secondary effects, in other words, would more than offset the primary effects. But such an outcome could be expected only under the rather unusual circumstances of high propensities to spend in a considerable number of the other countries.

VI. TWO-COUNTRY AND MULTIPLE-COUNTRY MODELS COMPARED

The classical theory of international trade, including the theory of comparative advantage as well as the closely-related theory of the international price mechanism, was developed almost entirely in terms of two countries. Most of the important problems in international economics during the nineteenth century were discussed as though the world economy were divided into two regions, one region being the home country—usually England—and the other region being the "rest of the world." During the interwar period of the present century, this classical procedure came under heavy attack, particularly by the late Professor Graham, who argued with considerable cogency and force that the classical procedure involved a persistent bias.[13] Graham insisted that the traditional, two-country theory greatly exaggerated the role of international demand and neglected the role of shifts in output in determining the terms of international exchange. He argued, specifically, that if one considers a complex world economy in which a large number of countries are trading in a considerable number of commodities, the process of adjustment to a disturbing event in international trade is fundamentally similar to the process of adjustment within a single country. In Graham's view, then, the fact that resources, particularly labor, are more or less immobile between countries does not require, as the classical economists had supposed, a theory of *international* prices, separate and distinct from the theory of prices within a single country.

In concluding the present paper, which has dealt with an international theory of income rather than a theory of prices, there is no need to

[13] F. D. Graham, "The Theory of International Values Re-examined," *Quarterly Journal of Economics*, Vol. 38, November, 1923, pp. 54–86, and "The Theory of International Values," *ibid.*, Vol. 46, August, 1932, pp. 581–616. The ideas contained in these two articles were considerably elaborated in book form. (See F. D Graham, *The Theory of International Values*, Princeton: Princeton University Press, 1948, 349 pp.)

discuss at length the controversy between Graham and the classical economists. My purpose in raising the issue is not to try to settle it but to raise a similar issue with respect to the international theory of income. If it is true, as Graham argues, that the traditional two-country model of international *price* theory involves a persistent and significant bias, is it also true that an analogous two-country model of international *income* theory involves a similar bias? To put the question another way, is a theory of international income that is founded upon the simplifying assumption that the world economy consists of two regions likely to involve any fundamental errors? The two-country income model, as I indicated earlier, has been discussed by a number of economists, and it should be possible to answer the question raised above by comparing the results of the two-country analysis with those of the generalized theory presented here. Since I am most familiar with my own version of the two-country model, I shall employ it to make the comparison.[14]

On the whole, the comparison does not reveal any basic flaws in the two-country model.[15] There are no processes of income adjustment in the n-country model which are not also revealed by the simple two-country model, and in the main the conclusions reached by employing the latter are the same as those reached by employing the former. In my earlier paper, using a terminology slightly different from that used here, I considered altogether three different cases of the two-country model. The first, or "normal," case was one in which the marginal propensity to spend was less than unity in both countries. The analogue of this case for the n-country model is the situation in which the propensity to spend is less than unity in each of the n countries. Under these circumstances both models reveal that an autonomous increase of investment in one country creates a deficit in that country's balance of payments on current account and that the amount of the deficit is less than the autonomous investment. This conclusion of the two-country model, in other words, is in no way vitiated by the complex interactions of trade among a large number of countries. The second case, in the two-country model, was one in which the propensity to spend of the country initiating the expansion, say Country 1, was less than unity, while the propensity to spend of the second country was greater than unity. The analogous situation, in the n-country model, is that in which the propensity to spend is less than unity in Country 1 but greater than unity in all other countries. Again, both the two-country and the n-country models lead to the same conclusion: autonomous investment in Country 1 actually *improves* the balance of trade of that country; the induced rise of Country 1's exports exceeds the rise of its imports. The third and final case, in the two-country model,

[14] Metzler, "Underemployment Equilibrium in International Trade," *op. cit.*
[15] Cf. Machlup, *op. cit.*, p. 197.

was a situation in which the propensity to spend in Country 1 was greater than unity, while the propensity to spend in Country 2 was less than unity; and the analogue of this situation, in the n-country model, is the situation in which Country 1 has a propensity to spend greater than unity, while all other countries have propensities less than unity. In this case also, as in the two preceding ones, the results of the two-country model are consistent with those of the n-country model. Either model supports the conclusion that, under the assumed conditions with respect to the propensities to spend, an increase in autonomous investment in Country 1 leads to an unfavorable movement in that country's balance of trade, the amount of the unfavorable movement being greater than the amount of autonomous investment.

Considering the large measure of agreement between the two-country and the n-country models, the reader may wonder what purpose is served by studying the generalized theory at all. If the simple theory and the general one both lead to the same results, why bother with the latter? To this question a number of answers may be given. The first and most obvious one is that hindsight is better than foresight. While we might have felt intuitively that the two-country model is satisfactory for most purposes, I doubt whether we could have been sure of this without a careful study of the more general system. A second reason for studying the general theory is that there are certain situations in the n-country model for which no analogue exists in the two-country model. This is true, for example, if the marginal propensity to spend is less than unity in Country 1 and in some but not all of the remaining countries. In situations such as this the effects of expansion can be described only by the general, n-country model. A third reason for preferring the n-country model to the two-country model is that the former provides a good deal more information than the latter about the dynamic stability of our income equations. Although I have used the stability conditions developed above primarily in studying the characteristics of the static equations, these stability conditions are also interesting and useful in other connections as well. It is useful, for example, to know that, if the propensity to spend is less than unity in all regions or subregions of the system, the stability of the income equations does not depend in any way upon how the world economy happens to be divided into national units. As a second example, it could easily be shown from Section III above and from my earlier paper on the stability of multiple markets that any cyclical solutions of the dynamic system are likely to be overshadowed by noncyclical solutions. This means, I believe, that the answer to the riddle of the business cycle is not to be found in horizontal transactions between one region and another, such as those depicted in our n-country system.

Perhaps the most important reason of all for studying the n-country model is that such a model will probably prove to be the most satisfactory theoretical foundation for an empirical study of the international aspects of income and employment. Although our study of the n-country model has not taken us very far, it has, I fear, taken us about as far as we can expect to go without introducing actual numbers in place of our hypothetical propensities to import and to spend. Unfortunately, the limits that we can expect to place upon the movements of our variables from a study of the theory alone are far too broad to be of much practical assistance in the formulation of economic policy. To a country considering the feasibility of a public works program, for example, it is little comfort to know that the unfavorable movement in its balance of trade engendered by such a program will normally be less than the amount of the public works. The country needs to know, in addition, what the approximate magnitude of its trade deficit will be and what the repercussions will be on incomes and trade balances in other countries. In order to answer questions such as these, it is obvious that the theory described above must be transformed into an empirical system; and for this purpose the n-country system is clearly the appropriate one. Eventually, then, an import-export matrix, similar in many respects to Leontief's input-output matrix for a single country, must be developed for the world economy. Many of the facts needed for such a table are already at hand. Reasonably accurate figures are available, for example, regarding the network of world trade. If these trade figures are to be transformed into propensities to import and to spend, however, they must be supplemented by statistics of national income for each of the countries. Lack of such income statistics has been responsible, more than anything else, for our inability to provide the empirical counterpart of the international theory of income set out above. With the improvement in statistics throughout the world since the end of the war, it is to be hoped that this gap in our knowledge will soon be filled.

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THE MULTI-SECTOR MULTIPLIER[1]

By John S. Chipman

The concept of the multiplier is extended to an economy composed of a number of sectors, such as countries, regions, industries, classes, and functional groups, or individuals, firms, and governments. The economy should be divided to such an extent as to maximize the stability of the parameters. If all the participating groups have the same marginal propensity to spend, the conventional multiplier formula is obtained. Some properties of such a system are described—in particular, the conditions for dynamic stability; the effect of changes in relative prices is also introduced.

I. INTRODUCTION

It is the purpose of this paper to formulate a multiplier applicable to a multi-sector economy, or to a number of related economies, and to indicate briefly its uses in economic theory.[2]

Aggregative analysis, despite its great advantages, suffers from the weakness that it treats heterogeneous magnitudes as homogeneous variables. Aggregative behavior functions may conceal widely varying individual behavior functions throughout the economy. It is therefore necessary for many purposes to divide the economy into its interrelated parts.

[1] Revision of a paper presented December 28, 1949, under the title "The Multi-Part Multiplier," before the Econometric Society, New York City. The argument of Section I has been reformulated and Section VI has been added.

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[2] The multi-sector multiplier is a logical extension to more than two countries of the foreign-trade multiplier as developed by Machlup [13], Metzler [14], and Lange [9, pp. 232–237]. An extension to more than two countries was developed in Machlup's work [13, Chapter VI and Appendix B], but with very limited assumptions with regard to the similarity of the different countries' spending patterns. The multi-country multiplier was first formulated in a note by Samuelson [18, pp. 223-224], which came to my attention after the present paper was first written. The suggestion that multiplier analysis be used to analyze the relation between groups other than countries or regions was first made, it appears, by Lange [9, pp. 232–233]. The stability conditions derived in Section V below are an extension of those formulated by Metzler [14, pp. 102–104]. Since this paper was presented, a similar article has appeared by Goodwin [6], which I discuss elsewhere [3].

Differences in the marginal propensities to spend of different economic organisms are of two sorts: nonrandom and random. To the extent that differences among economic organisms' marginal propensities to spend are nonrandom, these organisms may be described as heterogeneous; they may be called homogeneous to the extent that these differences are purely random. The disaggregation of a heterogeneous group or sector into a number of homogeneous groups will tend to increase the stability of the parameters (the marginal propensities to spend). On the other hand, the disaggregation of a homogeneous group into smaller groups will not increase the stability of the parameters; in fact, such disaggregation, beyond a certain point, will reduce the stability of the parameters, owing to the law of large numbers. Diminishing returns to subdivision set in when the disaggregation is carried out beyond the point where differences among organisms' marginal propensities to spend partake more and more of a random nature.[3] There will therefore be an optimum degree of sub-

[3] Thus, it is clearly an advantage, from this standpoint, to distinguish among individuals belonging to different classes or income-groups, or residing in different regions, and among firms belonging to different industries.

In addition to differences in the properties of organisms, differences in the properties of commodities will also influence the optimum degree to which the economy should be divided. The greater the extent to which the disaggregation is carried out, the larger will be the number of sectors towards which each sector will have a separate marginal propensity to spend. If commodities or groups of commodities are complementary with one another, a sector's marginal propensity to buy any of these commodities will tend to be relatively stable; on the other hand, a sector's marginal propensity to buy individual commodities or groups of commodities will tend to be unstable if they are relatively substitutable for one another. Thus, the marginal propensity to buy food or to buy housing is more stable than the marginal propensity to buy a particular kind of food or a particular kind of housing. There is a further sense, therefore, in which sectors may be said to be homogeneous or heterogeneous: a group of organisms is relatively homogeneous or heterogeneous in this sense if the products of these organisms are relatively substitutable for one another or relatively complementary with one another from the point of view of other sectors. If a group of organisms is heterogeneous with respect to these organisms' marginal propensities to spend, it is quite possible for the products of these organisms to be substitutable for one another; if this is so, the division of this heterogeneous group into homogeneous groups will reduce the stability of *other* groups' marginal propensities to spend. For instance, the division of the meat industry into the beef industry and the pork industry will reduce the stability of consumers' marginal propensities to spend, since the marginal propensities to buy beef and to buy pork will be less stable, taken separately, than the marginal propensity to buy meat. Whenever substitutability is present, there will be a tendency for the components of a sector's marginal propensity to spend to fluctuate in opposite directions. There is therefore a zone within which it may not be possible to increase the stability of some parameters without decreasing the stability of others. Hence, a certain amount of disaggregation of sectors producing substitutable commodities and aggregation of sectors with different marginal propensities to spend may be unavoidable.

division into homogeneous groups which maximizes the stability of the parameters. For empirical purposes, however, other considerations, such as the efficient statistical estimation of the parameters and the difficulties and expense of numerical computation, must enter into the optimum degree of division.

In spite of the possibility of considerable variability of the parameters associated with too minute a subdivision of the economy, there will still be sufficient stability to warrant a purely conceptual treatment of a multi-sector economy divided into individuals and firms. Furthermore, to the extent that the variability of the parameters arises out of the fact that spending-income functions are nonlinear, little generality is lost in the theoretical formulation by treating these functions as linear, while the mathematical treatment is thereby kept as simple as possible.[4] Thus, in Section VII, we are justified in incorporating relative price changes into the analysis.

The most obvious application of the multi-sector multiplier is its use in the analysis of interregional and international income movements. The economy may also be divided into classes or income groups, urban and rural areas, and government and private sectors.[5] The latter subdivisions may also be made in interregional and international analysis: each region or country may be divided into a number of sectors, such as business, government, banking system, capitalists, workers, farmers, etc. Theoretically, any such combination is possible, though for empirical purposes calculation of the multi-sector multiplier may become unnecessarily laborious; empirical evaluation of the parameters also presents serious difficulties, especially in interregional multiplier analysis.[6]

Each of the n parts of an economy will have a marginal propensity to spend internally ("intraspend") and marginal propensities to spend in the other $n - 1$ sectors ("extraspend"). If the economy is divided into individuals and firms, these may all be considered as having zero marginal propensities to intraspend. Firms' "incomes" should be considered, for these purposes, as their total receipts; only individuals' incomes should be aggregated into national income.[7] Individuals' marginal propensities to extraspend are their marginal propensities to buy the goods and services of other individuals and firms. Firms' marginal

[4] Cf. Samuelson [19, Chapter X].

[5] For models of multiplier relationships between capitalists and workers, and between the government and private sectors of the economy, see Chipman [2]; cf. also Lange [9, pp. 232–237].

[6] In the absence of statistics on interregional trade, some approximation to the values of the parameters may possibly be obtained by means of adaptations of Professor Sargant Florence's "location factor" (cf. Vining [21, 22]).

[7] Retained profits may be regarded as paid to a special group entitled "earners of retained profits."

propensities to extraspend will be equal to the slopes of their input-output functions, that is, to the derivative with respect to the value of their output, of their imports of materials from other firms, and of their imports of services from workers and capitalists (wages and profits). The resemblance of this case to Professor Leontief's input-output model [**10**, **11**, **12**] is close; the Leontief model may be considered as a special case of the multi-sector multiplier in which the sectors are industries (workers and capitalists being considered as "industries") whose marginal propensities to extraspend are equal to their average propensities to extraspend (i.e., input-output functions are homogeneous as well as linear).

In Section II the static formulation of the multiplier is developed, according to which it is possible to determine the increment in any sector's receipts that will ultimately result from an autonomous expenditure in any sector or, alternatively, from an autonomous expenditure distributed over several sectors. Likewise, it is possible to determine the ultimate change in income that will result from any autonomous change in the distribution of income. In Section III the dynamic multiplier is formulated by the introduction of a uniform spending-income lag for each sector, and it is shown that the static multiplier is a special case of the dynamic multiplier. It is proved in Section IV that, if all the participating sectors have equal marginal propensities to spend, the multiplier for their total receipts reduces to the well-known aggregative multiplier formula.

In Section V the correspondence between the static and dynamic systems is analyzed. It is found that stability of the system (convergence of the dynamic system to the equilibrium of the static system) requires that the determinant of the static system of equations be positive. Dynamic stability requires also that the mean value of all participating sectors' marginal propensities to intraspend must be less than unity in absolute value and that, if all marginal propensities to extraspend are nonnegative, each marginal propensity to intraspend must be less than unity in absolute value. If all autonomous expenditure throughout the economy is nonnegative and if all marginal propensities to extraspend are nonnegative, it is shown that the multiplier, if stable, converges to the minimum value of one half for the whole economy. However, it is quite possible to have a stable negative multiplier when autonomous expenditures are all nonnegative, provided there are some negative marginal propensities to extraspend, *even though each sector's aggregative marginal propensity to spend is positive;* such negative marginal propensities to extraspend may exist if some commodities are inferior goods and if governments attempt to stabilize national income by varying their expenditures inversely with their revenues. It is also possible for the

dynamic system to be unstable while the determinant of the corresponding static system is positive. Hence, the "correspondence principle" enunciated by Professor Samuelson does not always hold.

Necessary and sufficient conditions exist for the convergence of the dynamic system to equilibrium. These are stated in Section VI. Finally, in Section VII, changes in relative prices are brought into the analysis. It is assumed that the price or price index of a sector's products is a function of its income and that reactions to price changes take place after a uniform time lag. The effect of this refinement is to add a price-adjustment factor to each marginal propensity to spend.

II. THE STATIC MULTIPLIER

Let an economy be divided into n sectors, 1, 2, $\cdots$, n. Designating Y_i for the income of sector i, and E_i for expenditure in sector i, we may write ${}_iE'_j = \partial E_j/\partial Y_i$ for sector i's marginal propensity to spend in sector j. Let $dY_1, dY_2, \cdots, dY_n$ be the ultimate increments in income in sectors 1, 2, $\cdots$, n, respectively, resulting from the autonomous increments in expenditure $dE_1, dE_2, \cdots, dE_n$. Then for each sector the final increment in income consists of (1) the autonomous increment in expenditure dE_i, (2) the internally induced increment in expenditure ${}_iE'_i\,dY_i$, and (3) the externally induced increments in expenditure $\sum_{j=1}^{n} {}_jE'_i\,dY_j$ $(j \neq i)$. Thus,

$$
(1)\qquad
\begin{aligned}
dY_1 &= dE_1 + {}_1E'_1\,dY_1 + {}_2E'_1\,dY_2 + \cdots + {}_nE'_1\,dY_n\,,\\
dY_2 &= dE_2 + {}_1E'_2\,dY_1 + {}_2E'_2\,dY_2 + \cdots + {}_nE'_2\,dY_n\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
dY_n &= dE_n + {}_1E'_n\,dY_1 + {}_2E'_n\,dY_2 + \cdots + {}_nE'_n\,dY_n\,.
\end{aligned}
$$

These equations may be rewritten

$$
(2)\qquad
\begin{aligned}
&(1 - {}_1E'_1)dY_1 - {}_2E'_1\,dY_2 - \cdots - {}_nE'_1\,dY_n = dE_1\,,\\
&- {}_1E'_2\,dY_1 + (1 - {}_2E'_2)dY_2 - \cdots - {}_nE'_2\,dY_n = dE_2\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&{}_1E'_n\,dY_1 - {}_2E'_n\,dY_2 - \cdots + (1 - {}_nE'_n)dY_n = dE_n\,,
\end{aligned}
$$

or, in matrix form,

$$
(3)\qquad (I - M)dY = dE,
$$

where I is the identity matrix, M is the matrix

$$
(4)\qquad M = \begin{bmatrix} {}_1E'_1 & {}_2E'_1 & \cdots & {}_nE'_1 \\ {}_1E'_2 & {}_2E'_2 & \cdots & {}_nE'_2 \\ \cdots & \cdots & \cdots & \cdots \\ {}_1E'_n & {}_2E'_n & \cdots & {}_nE'_n \end{bmatrix}
$$

and where dY and dE are the column vectors

$$dY = \begin{bmatrix} dY_1 \\ dY_2 \\ \cdot \\ \cdot \\ \cdot \\ dY_n \end{bmatrix}, \qquad dE = \begin{bmatrix} dE_1 \\ dE_2 \\ \cdot \\ \cdot \\ \cdot \\ dE_n \end{bmatrix} \tag{5}$$

From (3) it follows that the multiplier may be written

$$dY = (I - M)^{-1} dE. \tag{6}$$

Writing Δ for the determinant of $I - M$, and $d\Delta_i$ for the determinant of the matrix formed by substituting dE in the ith column of $I - M$, we have the equation showing the income generated in the ith sector of the economy:

$$dY_i = d\Delta_i / \Delta. \tag{7}$$

The total income generated by dE in the whole economy is given by[8]

$$\sum_{i=1} dY_i = \frac{\sum_{i=1}^{n} d\Delta_i}{} \tag{8}$$

Let dA be an autonomous disturbance in expenditure in the whole economy, where $dE_i = e_i dA$. Then we may factor dA out of $d\Delta_i$ so that $d\Delta_i/dA = \Delta_i'$, where Δ_i' is formed by replacing the column vector dE of $d\Delta_i$ by e, where e is the column vector

(9)

$$e_n$$

Then the multiplier for the ith sector of the economy is

$$\frac{dY_i}{dA} = \frac{\Delta_i'}{\Delta}, \tag{10}$$

and for the economy as a whole,

$$\sum_{i=1}^{n} \frac{dY_i}{dA} = \frac{\sum_{i=1}^{n} \Delta_i'}{\Delta}. \tag{11}$$

[8] Firms and governments, of course, must not be included in the summation for national income.

Two kinds of autonomous disturbances in expenditure may be distinguished: (1) autonomous net change in expenditure in the economy, and (2) autonomous transfer of expenditure from one sector or group of sectors of the economy to another. In the first case, $dA = \sum_{i=1}^{n} dE_i$, and $\sum_{i=1}^{n} e_i = 1$. In the second case, $dA = \sum_{i=1}^{n} dE_i$ (for all $dE_i > 0$), and $\sum_{i=1}^{n} e_i = 0$. In the special case in which $e_1 = 1$ and all the other e_i's are zero, (7) and (10) become (for sector 1)

$$\frac{dY_1}{dE_1} = \frac{1}{1 - {}_1E_1' - \sum_{i=2}^{n} (C_i/C_1)\, {}_1E_i'}, \tag{12}$$

where C_i is the cofactor of the ith element in the first column of $I - M$. Other special cases are derived by giving various forms to the vector e. Thus, the effect on sector 1's income of expenditure in sector 2 is given by setting $e_2 = 1$ and all the other e_i's $= 0$; the effect on 1's income of an export from 1 to 2 is given by setting $e_1 = 1$, $e_2 = -1$, and all the other e_i's $= 0$; similarly, the effect of an export from 2 to 3 on 1's income is given by setting $e_2 = 1$, $e_3 = -1$, and all the other e_i's $= 0$. If the sectors are regions, the effect on 1's income of an autonomous expenditure in 1 that is partly devoted to expenditure on imported materials and profits accruing to the other $n - 1$ regions is evaluated by assigning the appropriate values to the e_i's, where $\sum_{i=1}^{n} e_i = 1$. The effect of any autonomous redistribution of income on income as a whole is evaluated by substituting the appropriate values for the e_i's in (11), where $\sum_{i=1}^{n} e_i = 0$.

III. THE DYNAMIC MULTIPLIER

We shall now assume that there is a uniform time lag of all sectors' expenditures behind their incomes.[9] Consequently, the change in any sector's income at time t will be a function of all sectors' incomes in the previous period $t - 1$. The system of equations (1) may then be transformed into the system of difference equations:

$$\begin{aligned}
dY_1(t) &= dE_1 + {}_1E_1'\, dY_1(t-1) + {}_2E_1'\, dY_2(t-1) + \cdots \\
&\qquad + {}_nE_1'\, dY_n(t-1), \\
dY_2(t) &= dE_2 + {}_1E_2'\, dY_1(t-1) + {}_2E_2'\, dY_2(t-1) + \cdots \\
&\qquad + {}_nE_2'\, dY_n(t-1), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
dY_n(t) &= dE_n + {}_1E_n'\, dY_1(t-1) + {}_2E_n'\, dY_2(t-1) + \cdots \\
&\qquad + {}_nE_n'\, dY_n(t-1),
\end{aligned} \tag{13}$$

[9] The meaning of this assumption is discussed in Chipman [3], where a device is introduced to take account of unequal time lags.

or, in matrix form,

$$IdY(t) - MdY(t-1) = dE, \tag{14}$$

where $dY(t)$ and $dY(t-1)$ are the column vectors $\{dY_i(t)\}$ and $\{dY_i(t-1)\}$ $(i = 1, 2, \cdots, n)$. The dynamic multiplier may therefore be written

$$dY = (I - M^t)(I - M)^{-1}dE. \tag{15}$$

The stationary part of the solution of (14) is given by the equilibrium condition

$$dY(t) = dY(t-1) = dY. \tag{16}$$

This reduces (14) to

$$IdY - MdY = dE, \tag{17}$$

which is the same as (3) and whose solution is therefore (7). The dynamic part of the solution is given by the characteristic equation of (14):

$$IdY(t) - MdY(t-1) = 0. \tag{18}$$

Let λ be the operator such that $\lambda dY(t-1) = dY(t)$. Then (18) becomes

$$(I\lambda - M)dY(t-1) = 0 \qquad [dY(t-1) \neq 0], \tag{19}$$

which can be true only if $I\lambda - M$ is singular. Consequently,

$$|I\lambda - M| = \begin{vmatrix} \lambda - {}_1E_1' & -{}_2E_1' & \cdots & -{}_nE_1' \\ -{}_1E_2' & \lambda - {}_2E_2' & \cdots & -{}_nE_2' \\ \cdots & \cdots & \cdots & \cdots \\ -{}_1E_n' & -{}_2E_n' & \cdots & \lambda - {}_nE_n' \end{vmatrix} = 0, \tag{20}$$

which is the characteristic polynomial of the system of equations (13). In its explicit form, (20) is written

$$\lambda^n - D_1\lambda^{n-1} + D_2\lambda^{n-2} - \cdots + (-1)^{n-1}D_{n-1}\lambda + (-1)^nD_n = 0, \tag{21}$$

where D_n is the determinant of M and D_i is the sum of the $n!/i!(n-i)!$ ith-order minors symmetric about the principal diagonal of M. Equations (13) therefore contract into the nth-order difference equation for sector i:

$$\begin{aligned} dY_i(t) - D_1\,dY_i(t-1) + D_2\,dY_i(t-2) - \cdots \\ + (-1)^nD_ndY_i(t-n) = d\Delta_i, \end{aligned} \tag{22}$$

and for all n sectors taken together:

$$\sum_{i=1}^{n} dY_i(t) - D_1 \sum_{i=1}^{n} dY_i(t-1) + D_2 \sum_{i=1}^{n} dY_i(t-2) - \cdots \tag{23}$$
$$+ (-1)^n D_n \sum_{i=1}^{n} dY_i(t-n) = \sum_{i=1}^{n} d\Delta_i,$$

where[10]

$$1 - D_1 + D_2 - \cdots + (-1)^n D_n = \Delta. \tag{24}$$

The general solution of (22) is therefore

$$dY_1 = \frac{d\Delta_i}{\Delta} + \sum_{j=1}^{n} k_{ij}\lambda_j^t, \tag{25}$$

and of (23),

$$\sum_{i=1}^{n} dY_i = \frac{\sum_{i=1}^{n} d\Delta_i}{\Delta} + \sum_{i=1}^{n}\sum_{j=1}^{n} k_{ij}\lambda_j^t, \tag{26}$$

where the k_{ij}'s are constants[11] and the λ_j's are the roots of (20).

If $|\lambda_j| < 1$ $(j = 1, 2, \cdots, n)$, then (25) and (26) converge to (7) and (8), respectively, as $t \to \infty$. If all the λ_j's are zero, (25) and (26) are equivalent to (7) and (8), respectively; likewise, if λ is the null operator, (22) and (23) also reduce to (7) and (8), respectively, by virtue of the expansion (24). This will be the case when reactions are instantaneous.[12] Thus the static multiplier is a special case of the dynamic multiplier.

IV. THE CONVENTIONAL MULTIPLIER

In this section we shall prove the following theorem.

THEOREM 1: *When all sectors' total marginal propensities to spend are equal to one another, the multi-sector multiplier reduces to the well-known conventional multiplier formula.*

PROOF: Let each sector's total marginal propensity to spend, $\sum_{j=1}^{n} {}_iE_j'$ (that is, each column sum of M), be equal to c. Then

$$(1, 1, \cdots, 1)M = (c, c, \cdots, c) \tag{27}$$

[10] If the equilibrium condition $dY_i = dY_i(t - \tau)$, $\tau = 0, 1, \cdots, n$, is satisfied, (22) clearly reduces to (7), owing to the expansion (24). Likewise, (23) reduces to (8).

[11] For the evaluation of these arbitrary constants, see Chipman [3].

[12] It will also be true in the trivial case in which $M = 0$, so that $dY_i = dE_i$ and $\Sigma_{i=1}^{n} dY_i = \Sigma_{i=1}^{n} dE_i$.

and

$$(1, 1, \cdots, 1)M^2 = (c, c, \cdots, c)M = (c^2, c^2, \cdots, c^2). \tag{28}$$

Thus it is seen that

$$(1, 1, \cdots, 1)M^t = (c^t, c^t, \cdots, c^t). \tag{29}$$

Now from (15), (9), and the definition of dA, it follows that the multiplier vector may be written

$$k = dY/dA = (I - M^t)(I - M)^{-1}e \tag{30}$$

so that

$$(I - M^t)e = (I - M)k. \tag{31}$$

Since each column sum of $I - M$ is equal to $1 - c$, and since, from (29), each column sum of $I - M^t$ is equal to $1 - c^t$, the column sums of the vectors (31) are

$$(1 - c^t)\sum_{i=1}^{n} e_i = (1 - c)\sum_{i=1}^{n} k_i \tag{32}$$

so that

$$\sum_{i=1}^{n} \frac{dY_i}{dA} = \sum_{i=1}^{n} k_i = \sum_{i=1}^{n} e_i \frac{1 - c^t}{1 - c} = \sum_{i=1}^{n} e_i \frac{1 - \left(\sum_{j=1}^{n} {}_iE'_j\right)^t}{1 - \sum_{j=1}^{n} {}_iE'_j}. \tag{33}$$

Thus, if $\sum_{i=1}^{n} e_i = 0$, $\sum_{i=1}^{n} k_i = 0$, so that any autonomous redistribution of income leaves aggregate income unchanged. If $\sum_{i=1}^{n} e_i = 1$, then $\sum_{i=1}^{n} k_i = (1 - c^t)/(1 - c)$, which is the familiar multiplier formula. The theorem is therefore proved.

V. THE CORRESPONDENCE BETWEEN THE STATIC AND DYNAMIC MULTIPLERS[13]

In this section we shall examine the relationship between the static and dynamic multi-sector multipliers and formulate necessary conditions for the stability of the latter in terms of certain properties of the former.

Stability requires that the roots of (20) lie within the unit circle of the complex plane: the absolute value of the modulus of every root must therefore be less than unity. We may relate the values of the roots to properties of the matrix M by means of the property that the ith-order

[13] I am greatly indebted to Professor A. H. Clifford for mathematical assistance in this section.

trace of a matrix is equal to the sum of the products of its latent roots taken i at a time.[14] Two of these traces, the first-order trace (or simply, the trace) and the nth-order trace (the determinant of M), provide us with necessary stability conditions.

CONDITION 1: *A necessary condition for stability is that the trace of $I - M$ be positive and less than twice the number of sectors.*

This follows from the fact that

$$\text{tr } M = D_1 = \sum_{i=1}^{n} {}_iE'_i = \sum_{i=1}^{n} \lambda_i, \tag{34}$$

so that stability requires that

$$-n < \text{tr } M < n; \tag{35}$$

that is to say, the sum of all the sectors' marginal propensities to intra-spend, being equal to the sum of the roots, must be less in absolute value than the number of sectors. The average marginal propensity to intra-spend must therefore be less than 1 and greater than -1.

CONDITION 2: *A necessary condition for stability is that the determinant of $I - M$ be positive.*

This follows from the fact that the determinant of M is equal to the product of its latent roots. This is demonstrated in the following theorem.

THEOREM 2: *If the real characteristic roots of M are less than unity, the determinant of $I - M$ is positive.*

PROOF: The determinant of $I - M$ may be reduced to the triangular form

$$| I - P^{-1}MP | = \begin{vmatrix} 1-\lambda_1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1-\lambda_2 & 0 & \cdots & 0 \\ -m_{31} & -m_{32} & 1-\lambda_3 & \cdots & 0 \\ \cdot\cdot & & & & \\ -m_{n1} & -m_{n2} & -m_{n3} & \cdots & 1-\lambda_n \end{vmatrix} \tag{36}$$

and is therefore expressible as

$$\Delta = \prod_{i=1}^{n} (1 - \lambda_i), \tag{37}$$

which is obviously positive if every $\lambda_i < 1$. If there are any complex characteristic roots, each complex root λ has its conjugate complex root $\bar{\lambda}$. Hence, each complex $1 - \lambda$ must be multiplied by its conjugate $1 - \bar{\lambda}$, the product being positive, for

$$(1 - \lambda)(1 - \bar{\lambda}) = | 1 - \lambda |^2 > 0. \tag{38}$$

[14] Cf. Aitken [1, p. 88].

Complex roots can therefore not make Δ negative. The theorem is therefore proved.

The conditions $-n < \operatorname{tr} M < n$ and $\Delta > 0$ are both necessary conditions for stability, but they are not sufficient conditions.[15] If either the determinant or the trace of $I - M$ is negative, we know that the system is monotonically unstable. It is still possible, however, for the system to be unstable while the two stability conditions hold. For instance, we know that whenever M has negative or complex characteristic roots and an even number of roots greater than unity, $\Delta > 0$. Thus, there may be either monotonic or oscillatory instability without our two conditions reflecting them.

We shall now prove that when there is no autonomous transfer of expenditure from one sector of an economy to another, and when no marginal propensity to extraspend is negative, the multiplier for the whole economy must be positive. Let each $dE_i \geqslant 0$ and at least one $dE_i > 0$; or, alternatively, let each $dE_i \leqslant 0$ and at least one $dE_i < 0$. Then the vector e may be of any conceivable form provided it has no elements of opposite sign and $\sum_{i=1}^{n} e_i = 1$. Then, if the total multiplier $\sum_{i=1}^{n} dY_i / \sum_{i=1}^{n} dE_i$ is to be positive, the column sums of $(I - M)^{-1}$ must all be positive; for, if any column sum of $(I - M)^{-1}$ (say the kth) is not positive, there will exist a vector e (namely, the one in which $e_k = 1$ and all the other e_i's are zero) which will make the multiplier nonpositive. Since stability requires that $\Delta > 0$, the condition that the column sums of $(I - M)^{-1}$ all be positive requires, under stable conditions, that the column sums of adj $(I - M)$ all be positive. The following theorem, which is due in its entirety to Professor Clifford, proves that the column sums of adj $(I - M)$ are positive if marginal propensities to extraspend are not negative and the economy is stable.

THEOREM 3: *Let A be an $n \times n$ matrix satisfying the following two conditions:* (a) *the off-diagonal elements of A are $\geqslant 0$;* (b) *A has no real characteristic root $\geqslant 1$. Then the column sums of adj $(I - A)$ are all positive.*

PROOF: Let $B(t) = I - tA$, where t is a real variable. Then $B(0) = I$, $B(1) = I - A$, and, by Condition (b),

$$\det B(t) > 0 \quad (0 \leqslant t \leqslant 1). \tag{39}$$

For, were $\det B(t_1) = 0$, $1/t_1$ would be a characteristic root of A.

Let $c_1(t), c_2(t), \cdots, c_n(t)$ be the column sums of adj $B(t)$. They are polynomials in t all having the value 1 when $t = 0$. We are to show that they are all positive when $t = 1$. If they are not, then let t_1 be the

[15] When $n = 2$, they are also sufficient conditions for monotonic stability; for, if one root exceeds unity, $\Delta < 0$, whereas if both exceed unity, $\operatorname{tr} M > 2$. This has been shown by Metzler [14, p. 102, equation (7)], under the implicit assumption that the propensities are positive.

smallest positive root of $c_i(t)$ less than or equal to 1, or define $t_1 = 1$ if $c_i(t)$ has no real root in the interval (0, 1). Let t_0 be the least of the real numbers $t_1, t_2, \cdots, t_n$. Then at least one $c_i(t)$ vanishes for $t = t_0$, while the remaining ones are positive at $t = t_0$.

Now observe that

$$[c_1(t), c_2(t), \cdots, c_n(t)] = (1, 1, \cdots, 1) \text{ adj } B(t); \tag{40}$$

hence,

$$[c_1(t), c_2(t), \cdots, c_n(t)]\, B(t) = [\det B(t), \det B(t), \cdots, \det B(t)]. \tag{41}$$

Written out, we have

$$\sum_{i=1}^{n} c_i(t)\, b_{ij}(t) = \det B(t) \qquad (j = 1, 2, \cdots, n), \tag{42}$$

where $b_{ij} = \delta_{ij} - ta_{ij}$.

For simplicity, let us suppose that $c_1(t)$ is one of the $c_i(t)$ which vanish at $t = t_0$. Hence,

$$c_1(t_0) = 0, \qquad c_i(t_0) \geqslant 0 \qquad (i = 2, 3, \cdots, n). \tag{43}$$

Consider the first equation in (42):

$$c_1(t)\,(1 - ta_{11}) - c_2(t)ta_{21} - \cdots - c_n(t)ta_{n1} = \det B(t). \tag{44}$$

Set $t = t_0$ in this, and note that t_0 is positive, $c_i(t_0) \geqslant 0$ for $i \neq 1$, and $a_{i1} \geqslant 0$ for $i \neq 1$ by Condition (a). Hence, $c_i(t_0)t_0a_{i1} \geqslant 0$ for $i \neq 1$. But $c_1(t_0) = 0$. Hence the left-hand member is $\leqslant 0$, while by (39) the right-hand member is >0. We arrive thereby at a contradiction. The theorem is therefore proved.

This theorem shows us that negative marginal propensities to intraspend cannot make the multiplier negative. It may be verified by a simple example that negative marginal propensities to extraspend may bring about negative column sums of adj $(I - M)$ and consequently a negative multiplier. Negative marginal propensities to extraspend will be characteristic of an economy some of whose sectors are in the habit of buying inferior goods from other sectors, and in which governments react to changes in their revenues by changing their expenditures in the opposite direction (in order, say, to stabilize national income). Let

$$M = \begin{bmatrix} 1.1 & 0.1 \\ -0.3 & 0.8 \end{bmatrix},$$

so that

$$I - M = \begin{bmatrix} -0.1 & -0.1 \\ 0.3 & 0.2 \end{bmatrix}, \quad \text{tr } M = 1.9, \quad \Delta = 0.01,$$

and

$$(I - M)^{-1} = \begin{bmatrix} 20 & 10 \\ -30 & -10 \end{bmatrix}.$$

Then the column sums of $(I - M)^{-1}$ are respectively -10 and 0, so that the multiplier for the whole economy is $-10e_1$. This example shows that the negative marginal propensity to extraspend is accompanied by a greater than unit marginal propensity to intraspend. Thus, the multiplier may be stable and negative even though the aggregate marginal propensity to spend is positive in each part of the economy. The multiplier is positive provided only that marginal propensities to extraspend are positive.

It is therefore possible to have a normal static multiplier corresponding to instability and an abnormal static multiplier compatible with stability. Consequently, Professor Samuelson's "correspondence principle" does not always apply.[16]

We shall now develop a further stability condition. First we shall prove the following theorem:

THEOREM 4: *The order of a nonsingular matrix is equal to the sum of the products of its ith-row sum and the ith-column sum of its inverse.*

PROOF: Let r_i be the ith-row sum of an $n \times n$ matrix A, and let c_i' be the ith-column sum of A^{-1}. Then

$$(1, 1, \cdots, 1)A^{-1} = (c_1', c_2', \cdots, c_n'), \tag{45}$$

so that

$$(1, 1, \cdots, 1) = (c_1', c_2', \cdots, c_n')A. \tag{46}$$

Taking the row sums of both sides of (46), we obtain

$$n = c_1'r_1 + c_2'r_2 + \cdots + c_n'r_n, \tag{47}$$

which was to be proved.

Applying this theorem to the matrix $(I - M)$, we obtain from (46) the following equations:

$$\begin{aligned} 1 &= c_1'(1 - {}_1E_1') + c_2'(-{}_1E_2') + \cdots + c_n'(-{}_1E_n'), \\ 1 &= c_1'(-{}_2E_1') + c_2'(1 - {}_2E_2') + \cdots + c_n'(-{}_2E_n'), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1 &= c_1'(-{}_nE_1') + c_2'(-{}_nE_2') + \cdots + c_n'(1 - {}_nE_n'). \end{aligned} \tag{48}$$

[16] The correspondence principle, as defined by Samuelson [18, p. 225], states, "Systems which are well-behaved dynamically tend to be well-behaved from a purely comparative statical viewpoint; likewise those which are unstable give rise to anomalous comparative statical conclusions."

Let all marginal propensities to extraspend be $\geqslant 0$; then each $c_i' > 0$. Hence, the larger the marginal propensities to extraspend, the smaller the r_i's, the larger the c_i''s, and the larger the multiplier. Large marginal propensities to extraspend cannot make the multiplier less unstable than small ones. Consequently, any necessary conditions for stability that apply when marginal propensities to extraspend are zero must also apply when they are positive. Let us assume, therefore, that all of the off-diagonal elements of M are zero. Then it can easily be shown that, as is to be expected,

$$(I - M^t)(I - M)^{-1} = \begin{bmatrix} \frac{1 - {}_1E_1'^{\,t}}{1 - {}_1E_1'} & 0 & \cdots & 0 \\ 0 & \frac{1 - {}_2E_2'^{\,t}}{1 - {}_2E_2'} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1 - {}_nE_n'^{\,t}}{1 - {}_nE_n'} \end{bmatrix}, \tag{49}$$

that is, each sector's closed system multiplier is entered in the principal diagonal. Now, each of these closed system multipliers approaches the minimum stable value of $\frac{1}{2}$;[17] consequently, the multiplier for the whole economy approaches the same value as a minimum. Therefore, as long as marginal propensities to extraspend are not negative and the economy is stable, the multiplier must not only be positive but it must exceed the value $\frac{1}{2}$.

Equation (49) also tells us that stability requires that *each* ${}_iE_i'$ be less than unity in absolute value. Now, if marginal propensities to extraspend are positive, the multiplier will be larger, and this stability condition will be reinforced. Thus, we arrive at the important proposition that not only must the average marginal propensity to intraspend, tr M/n, be less than unity, but, *provided every marginal propensity to extraspend is nonnegative, every marginal propensity to intraspend must be less than unity*. In mathematical terms, this means that if the characteristic roots of a matrix are less than unity, and if the off-diagonal elements of the matrix are nonnegative, then each diagonal term must be less than unity. In terms of the matrix $I - M$, if each off-diagonal term is nonpositive, each diagonal term must be positive.

[17] As ${}_iE_i'$ approaches -1, $\lim_{t\to\infty} (1 - {}_iE_i'^{\,t})/(1 - {}_iE_i')$ approaches the value $\frac{1}{2}$; since stability requires that ${}_iE_i' > -1$, $\frac{1}{2}$ is the minimum value. If negative marginal propensities to intraspend are ruled out, the minimum value of the multiplier is 1.

We may therefore draw from this section the following conclusions:

(1) A necessary condition for stability is that the sum of the marginal propensities to intraspend must be less in absolute value than the number of sectors.

(2) A necessary condition for stability of the static multiplier is that its determinant be positive.

(3) If no autonomous expenditures are of opposite sign and if there are no negative marginal propensities to extraspend, the multi-sector multiplier must exceed $\frac{1}{2}$ in value.

(4) If there are no negative marginal propensities to extraspend, a necessary condition for stability is that each marginal propensity to intraspend be less than unity.

VI. THE NECESSARY AND SUFFICIENT CONDITIONS FOR STABILITY

The necessary and sufficient conditions for the stability of the system of difference equations (13) are that the roots of the characteristic polynomial (21) lie within the unit circle of the complex plane.[18] We shall set forth, in the present section, the necessary and sufficient conditions, which may be called the Schur conditions, that the roots of a polynomial lie within the unit circle. First, it will be convenient to summarize the development of necessary and sufficient stability conditions in the mathematical literature.[19]

The stability of a system described by a system of differential equations requires that the roots of its characteristic polynomial have all their real parts negative.[20] The necessary and sufficient conditions that the roots of a polynomial lie in the negative half of the complex plane were first developed by Routh [**15**, Chapters II and III; **16**, Chapter VI, paragraphs 290–307]. Later they were derived independently by Hurwitz [**8**] in a more convenient form involving determinants. More recently

[18] We shall not consider here the peculiar case, which I discuss elsewhere [**3**], in which roots that are not within the unit circle may be annihilated by zero coefficients. In this rare and unlikely case, the necessary and sufficient conditions that the characteristic roots be less than unity in absolute value are not necessary conditions for stability; the further condition—the nonvanishing of the coefficients of the roots—must then be added in order to make the stability conditions necessary and sufficient.

[19] I am indebted to Professor Aurel Wintner, Professor of Mathematics at The Johns Hopkins University, for the references, given below, to the articles of Hurwitz, Cohn, and Herglotz.

[20] The solution of a system of differential equations is in terms of functions of the form $e^{\omega t}$, where ω is a complex root of the form $\mu + \nu i$. By Taylor's theorem, $e^{\omega t} = e^{\mu t}(\cos \nu t + i \sin \nu t)$; convergence therefore requires that $\mu < 0$. In the case of difference equations the solution is in terms of functions of the form λ^t, where $\lambda = e^{\omega}$, $|\lambda| = e^{\mu}$, and $\arg \lambda = \nu$. Hence, the condition $\mu < 0$ is the same as the condition $|\lambda| < 1$.

Frazer and Duncan [5], unaware of Hurwitz' work, reformulated the Routh conditions in the Hurwitz form.[21]

The necessary and sufficient conditions that the roots of a polynomial lie within the unit circle of the complex plane were first derived by Schur [20] and developed by Cohn [4]. It was subsequently shown by Herglotz [7] that the Cohn conditions are equivalent to the Hurwitz conditions. The one may be transformed into the other by means of a transformation given by Schur [20, p. 288]. The same transformation was in recent years used by Samuelson [17] (who was unaware of the work of Schur, Cohn, and Herglotz) to derive, from the Routh-Hurwitz conditions, the necessary and sufficient conditions that the roots of a polynomial lie within the unit circle.

The Schur conditions are as follows [4, p. 125]: *The necessary and sufficient conditions that the roots of the polynomial*

$$f(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{50}$$

lie within the unit circle of the complex plane are that the n determinants

$$\delta_i = \left|\begin{array}{cccc|cccc} a_0 & 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_{n-i+1} \\ a_1 & a_0 & \cdots & 0 & 0 & a_n & \cdots & a_{n-i+2} \\ \vdots & & & & & & & \vdots \\ a_{i-1} & a_{i-2} & \cdots & a_0 & 0 & 0 & \cdots & a_n \\ \hline a_n & 0 & \cdots & 0 & a_0 & a_1 & \cdots & a_{i-1} \\ a_{n-1} & a_n & \cdots & 0 & 0 & a_0 & \cdots & a_{i-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-i+1} & a_{n-i+2} & \cdots & a_n & 0 & 0 & \cdots & a_0 \end{array}\right| \tag{51}$$

where $i = 1, 2, \cdots, n$, *all be positive definite.*[22] These conditions may readily be applied to the characteristic polynomial (20) by making the substitution $a_i = (-1)^i D_i$, where $a_0 = 1$.

Since the Schur conditions do not require any transformations but may be derived directly from the coefficients of the characteristic poly-

[21] A statement of the Hurwitz conditions may be found in Samuelson [19, pp. 433–434].

[22] In the original formulation the coefficients of the polynomial are taken as complex; hence, the two bottom partitioned matrices of δ_i are, in the complex case, not the transposes of the diagonally opposite partitioned matrices (as shown above), but the transposes of their complex conjugates. This more general case obviously does not concern us here.

It should be noted that a_0 in (50) is taken to be positive.

nomial, they are in a more convenient form than the Samuelson conditions.

VII. CHANGES IN RELATIVE PRICES

Up till now we have implicitly assumed that there are no changes in relative prices. We may now relax that assumption. The expenditure of sector i in sector j may be considered as a function not only of i's income but also of the prices (or price indices) of all sectors of the economy.[23] Denoting autonomous expenditure in j by E_j, k's spending in j by ${}_kE_j$, and the price of j's product (or price index of its products) as a function of j's income (receipts) by $p_j(Y_j)$, we may write for j's income equation:

$$\begin{aligned} Y_j = E_j &+ {}_1E_j[Y_1, p_1(Y_1), \cdots, p_n(Y_n)] \\ &+ {}_2E_j[Y_2, p_1(Y_1), \cdots, p_n(Y_n)] + \cdots \\ &+ {}_nE_j[Y_n, p_1(Y_1), \cdots, p_n(Y_n)]. \end{aligned} \tag{52}$$

Thus,

$$\begin{aligned} dY_j = dE_j &+ \\ &\left({}_1E_{jY_1} + \sum_{k=1}^{n} {}_kE_{jp_1} p_1'\right) dY_1 + \left({}_2E_{jY_2} + \sum_{k=1}^{n} {}_kE_{jp_2} p_2'\right) dY_2 \\ &+ \cdots + \left({}_nE_{jY_n} + \sum_{k=1}^{n} {}_kE_{jp_n} p_n'\right) dY_n. \end{aligned} \tag{53}$$

That is to say, as j's income rises, j's price will rise, so that all sectors' expenditures in every other sector will change.

If changes in expenditure lag equally behind changes in income and changes in prices (i.e., ${}_kE_j(t) = {}_kE_j[Y_j(t-1), p_1(t-1), \cdots, p_n(t-1)]$) and if changes in prices follow changes in income instantaneously (i.e., $p_j(t) = p_j[Y_j(t)]$), then it is clear that (53) becomes

$$\begin{aligned} dY_j(t) = dE_j &+ \left({}_1E_{jY_1} + \sum_{k=1}^{n} {}_kE_{jp_1} p_1'\right) dY_1(t-1) \\ &+ \left({}_2E_{jY_2} + \sum_{k=1}^{n} {}_kE_{jp_2} p_2'\right) dY_2(t-1) + \cdots \\ &+ \left({}_nE_{jY_n} + \sum_{k=1}^{n} {}_kE_{jp_n} p_n'\right) dY_n(t-1). \end{aligned} \tag{54}$$

Thus we see that when income-induced price changes are taken into account, i's marginal propensity to spend in j becomes

$$ {}_iE_j' = {}_iE_{jY_i} + \sum_{k=1}^{n} {}_kE_{jp_i} p_i'. \tag{55}$$

[23] In order to remove money illusion, one of these prices may be taken as *numéraire* and set equal to unity.

We form the matrices

$$L = ({}_iE_{jY_i}) \tag{56}$$

and

$$K = \left(\sum_{k=1}^{n} {}_kE_{j_{p_i}}\right) \tag{57}$$

and the column vector

$$p' = \begin{bmatrix} p_1' \\ p_2' \\ \vdots \\ p_n' \end{bmatrix} \tag{58}$$

where the i subscript refers to columns and the j subscript to rows. The matrix M then becomes

$$M = L + Kp'. \tag{59}$$

The preceding analysis holds generally for the adjusted matrix M.

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IDENTIFIABILITY OF A LINEAR RELATION BETWEEN VARIABLES WHICH ARE SUBJECT TO ERROR[1]

By Olav Reiersøl

The problem of the identifiability of a linear relation between variables subject to error is examined in two different two-variable models in which the errors are independent of the "true" variables. In one model the errors are specified to be jointly normally distributed, and in the other to be stochastically independent. In both cases necessary and sufficient conditions for identifiability are found. A summary of previous results is included.

1. INTRODUCTION

Let $x_{1t}, x_{2t}, \cdots, x_{kt}$ $(t = 1, 2, \cdots, T)$ be observed variables and let y_{it} be the "true" value of x_{it}. The error v_{it} will be defined by

$$v_{it} = x_{it} - y_{it} \qquad (i = 1, 2, \cdots, k; t = 1, 2, \cdots, T). \tag{1.1}$$

We shall denote the vector $[x_{1t}, x_{2t}, \cdots, x_{kt}]$ by x_t, with similar interpretations of y_t and v_t. The subscript t may denote a point of time, but it may also denote any other ordering of the observations.

We shall assume that there exists a linear relation between the variables y_{it},

$$\sum_{i=1}^{k} \alpha_i y_{it} + \alpha_0 = 0, \tag{1.2}$$

which can be written in vector notation as

$$\alpha y_t' + \alpha_0 = 0. \tag{1.2'}$$

We shall consider different models, each of which contains specifications (1.1) and (1.2) and additional specifications. When all parameters and all distributions in the model are specified, we shall talk about a *structure*.[2] A structure is thus a particular realization of the model, and the model is the set of all structures compatible with the given specifications.

The variables x_{it} are *observed variables*. The variables y_{it} and v_{it} are not observed and are called *latent variables.*

A structure generates one and only one distribution $P(x)$ of the observed variables. On the other hand, there may be several structures generating the same distribution $P(x)$. If two or more structures generate

[1] This paper will be included in Cowles Commission Papers, New Series, No. 39.

[2] The terms defined here and in the rest of this introductory section are those used in publications of the Cowles Commission for Research in Economics. See, for instance, [2] and [12].

the same joint probability distribution of the observed variables, the structures are said to be *equivalent*. If a parameter has the same value in all equivalent structures it is said to be *identifiable*. In other words, a parameter is identifiable if it can be uniquely determined from a knowledge of the joint probability distribution of the observed variables. If a parameter is not identifiable, no consistent estimate of the parameter will exist. Conversely, if an estimate of a parameter has been proved to be consistent, the parameter must be identifiable.

Equation (1.2) remains the same if $\alpha_1, \cdots, \alpha_k$ and α_0 are all multiplied by the same constant. In order to make the coefficients determinate, we must have a normalization rule for these coefficients. In the following we shall tacitly assume that such a rule has been imposed.

More generally, we might consider models where equation (1.2) is replaced by a set of equations. Closely related to such models are the models considered in psychological factor analysis.

2. SURVEY OF PREVIOUS RESULTS

In this section I shall give a survey of some previous studies, formulated in modern terminology.

I shall not be concerned with the multiequational case or with factor analysis, except that I want to point out that discussions of identifiability in factor analysis date as far back as 1919, when G. H. Thomson [23] demonstrated by numerical examples the nonidentifiability of structures in a certain model of factor analysis.[3]

Gini [9] considered the particular case in which $k = 2$. He specifies that v_1 and v_2 are random variables which have zero correlations with x_1 and x_2, and that v_1 and v_2 are uncorrelated. (Two variables will be said to be "uncorrelated" or to have "zero correlation" if their covariance is equal to zero.) Gini shows that the parameter $\beta = -\alpha_2/\alpha_1$ is identifiable if we know the variance of v_1 or the variance of v_2 or the ratio between the two variances. He also shows that we get a bias in one direction if we replace the coefficient β by β_{12} and a bias in the opposite direction if we replace β by $1/\beta_{21}$, where β_{12} and β_{21} are coefficients in the least-squares linear regression of x_1 on x_2: $x_1 = \beta_{12}x_2 + \beta_{10}$, and the least-squares linear regression of x_2 on x_1: $x_2 = \beta_{21}x_1 + \beta_{20}$.

From Gini's results it follows immediately that β lies between the limits β_{12} and $1/\beta_{12}$. In this form the result was stated by Frisch [4, p. 60].[4] Frisch assumes that similar limits exist in the case of a linear

[3] References to other discussions of identifiability in factor analysis models are found in [21].

[4] It should be noted that Gini and Frisch do not consider a hypothetical universe but a sample; however, their considerations can immediately be translated into statements about a universe, or about a sample, the size of which tends to infinity.

relation between n variables which are all subject to error,[5] and he partly bases his confluence analysis on this assumption. This analysis gives methods which can be used whether α is identifiable or not, but if α is identifiable we can, of course, do something better than use the limits mentioned.

Koopmans [**11**, p. 70] considered a model given by specifications (1.1), (1.2), and the following specifications, (2.1)–(2.5).

(2.1) The vector variable $[y_t\ v_t]$ is independent of the vector variable $[y_{t'}\ v_{t'}]$ when $t \neq t'$.

(2.2) The distribution of $[y_t\ v_t]$ is independent of t.

Let $[y\ v]$ be a vector variable which has the same distribution as each of the variables $[y_t\ v_t]$.

(2.3) y is independent of v.

(2.4) The variables $v_1, v_2, \cdots, v_k$ are independent.

(2.5) The variables y and v are jointly normally distributed.

Koopmans points out that α is not identifiable in this model.

On the other hand, he indicates that α is identifiable in the case $k = 2$ if the vector y, instead of being normally distributed, can take only two values.

Wald [**27**] considered the case where $k = 2$ and where the variables x_{1t} can be divided into two groups $x_{11}, \cdots, x_{1m}$ and $x_{1,m+1}, \cdots, x_{1,2m}$ such that the limit inferior of $(1/m)|(y_{11} + \cdots + y_{1m}) - (y_{1,m+1} + \cdots + y_{1,2m})|$, when m tends to infinity, is positive. He also specifies that the distributions of v_{1t} and v_{2t} are independent of t, that $\mathrm{cov}(v_{it}v_{it'}) = 0$ when $t \neq t'$, $i = 1$ or 2, and that $\mathrm{cov}(v_1v_2) = 0$. Under these specifications, Wald shows that we can find a consistent estimate for the parameter α_2/α_1 and other parameters of the model. This implies that these parameters are identifiable.

The author of the present paper proved several theorems [**19**] which give methods to determine the vector α, or limits for this vector, in terms of the parameters of the probability distribution of the observed variables. Both kinds of theorems give as immediate corollaries statements about the identifiability of α.

In [**19**, Section 4] he considered the following model. The variables y_t and v_t represent stationary sets of time series. We suppose that y_t and v_{t+u} are uncorrelated and that v_t and v_{t+u} are uncorrelated, where $u \neq 0$. Let M_u be the square k-rowed matrix $[\mu_{ij.u}]$, where $\mu_{ij.u} = \mathrm{cov}(x_{it}x_{j,t+u})$.

[5] This assumption was later verified by Koopmans [**11**, p. 101].

In this model α is identifiable if M_u is of rank $k - 1$, i.e., if at least one element of the adjoint of M_u is different from zero. This statement follows as a corollary from [**19**, Theorem 4]. Other results concerning the identifiability of α follow from [**19**, Sections 2, 3, and 12].

In [**19**, Section 11], the author considered a model where there is an extra set of observed variables $x_{k+1}, x_{k+2}, \cdots, x_h$, such that x_j and v_i are uncorrelated when $i = 1, 2, \cdots, k$, and $j = k + 1, k + 2, \cdots, h$. Let $\mu_{ij} = \text{cov}(x_i x_j)$. Then we have

$$\sum_{i=1}^{k} \mu_{ij} \alpha_i = 0 \qquad (j = k + 1, \cdots, h). \tag{2.6}$$

If the rank of the matrix of this system of equations is $k - 1$, it is evident that the vector α is identifiable.

In a later paper [**20**] the author has called the set of variables $x_{k+1}, x_{k+2}, \cdots, x_h$ an instrumental set of variables. He gives several examples of sets of variables which may possibly be used as instrumental sets of variables.

In the meantime, the idea of using instrumental variables was introduced independently by Geary [**6**]. In another paper [**5**] Geary considers a model given by specifications (1.1), (1.2), and (2.1)–(2.4). Let $\kappa(c_1, c_2, \cdots, c_k)$ denote a semi-invariant of the joint distribution of $x_1, x_2, \cdots, x_k$ which is of degree c_i in the variable x_i ($i = 1, 2, \cdots, k$). Geary shows that the parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$ satisfy the following equation,

$$\sum_{i=1}^{k} \alpha_i \kappa(c_1, \cdots, c_{i-1}, c_i + 1, c_{i+1}, \cdots, c_k) = 0, \tag{2.7}$$

for each set of nonnegative integers $c_1, c_2, \cdots, c_k$ where at least two of the c's are positive. If the matrix of this system of equations is of rank $k - 1$, the vector α is identifiable.

Neyman and Scott [**18**, Section 8] considered the problem of finding consistent estimates of the parameters in the case where there is more than one observation for fixed i and t.

It should be noted that this survey is restricted to papers which directly or indirectly give new results concerning the identifiability of structures in a model where there is only one linear relation between the true variables. Some of the papers mentioned also consider problems of estimation and sampling distributions, and some of them consider models with several equations. Other papers giving contributions to these problems have been published by Geary [**7**, **8**], Tintner [**24**, **25**, **26**], Housner and Brennan [**10**], Neyman [**17**], Scott [**22**], Nair and Shrivastava [**16**], Nair and Banerjee [**15**], Lindley [**14**], and Bartlett [**1**]. In [**17**] Neyman

derives a consistent estimate of the parameter β in the case considered in Section 3 of the present paper.

3. CONDITIONS FOR THE IDENTIFIABILITY OF A LINEAR RELATION BETWEEN TWO VARIABLES WHEN THE JOINT DISTRIBUTION OF THE ERRORS IS NORMAL

In the rest of this paper we shall suppose that specifications (2.1) and (2.2) are satisfied, in other words, that each set of observations $[y_i v_i]$ can be considered as a random drawing from a probability distribution of the variables y and v. We shall furthermore restrict ourselves to the case when $k = 2$. Equations (1.1) and (1.2) now may be written in the form

$$x_i = y_i + v_i, \qquad (i = 1, 2) \tag{3.1}$$

$$y_2 = \beta y_1 + \beta_0 . \tag{3.2}$$

In order not to exclude the case $\alpha_2 = 0$, we shall consider $\beta = \infty$ and $\beta_0 = \infty$ as admissible values of β and β_0.

We shall introduce three further specifications:

(3.3) The set of variables v is independent of the set of variables y.

(3.4) The joint distribution of the errors v_1 and v_2 is normal.

$$\mathcal{E}(v_1) = \mathcal{E}(v_2) = 0. \tag{3.5}$$

The model defined by specifications (3.1)–(3.5) will be called Model A.

We shall use the letter ϕ for the characteristic function of a distribution. Characteristic functions for different distributions will be distinguished by putting the variables of the distribution as subscripts of the letter ϕ—for instance, $\phi_{x_1x_2}(t_1t_2) = \mathcal{E}(e^{ix_1t_1+ix_2t_2})$. The logarithm of a characteristic function ϕ will be denoted by ψ.

Let $\lambda_{11} = \text{var}(v_1)$, $\lambda_{22} = \text{var}(v_2)$, and $\lambda_{12} = \text{cov}(v_1v_2)$. Then

$$\phi_{v_1v_2}(t_1, t_2) = \exp\{-\tfrac{1}{2}(\lambda_{11}t_1^2 + 2\lambda_{12}t_1t_2 + \lambda_{22}t_2^2)\} \tag{3.6}$$

because of specifications (3.4) and (3.5). Using (3.2) we obtain

$$\phi_{y_1y_2}(t_1, t_2) = \exp\{\beta_0 it_2\}\phi_{y_1}(t_1 + \beta t_2). \tag{3.7}$$

From (3.6), (3.7), and specifications (3.1) and (3.3) we obtain

$$\phi_{x_1x_2}(t_1, t_2) = \exp\{\beta_0 it_2 - \tfrac{1}{2}(\lambda_{11}t_1^2 + 2\lambda_{12}t_1t_2 + \lambda_{22}t_2^2)\}\phi_{y_1}(t_1 + \beta t_2). \tag{3.8}$$

Suppose now that there exist two different structures,

$$S = \{\beta, \beta_0, \lambda_{11}, \lambda_{12}, \lambda_{22}, \psi_{y_1}(t)\}$$

and

$$S^* = \{\beta^*, \beta_0^*, \lambda_{11}^*, \lambda_{12}^*, \lambda_{22}^*, \psi_{y_1}^*(t)\},$$

which generate the same probability distribution $P(x)$ of the observed variables. Then we have

$$\begin{aligned}(3.9)\qquad &\exp\{\beta_0 it_2 - \tfrac{1}{2}(\lambda_{11}t_1^2 + 2\lambda_{12}t_1t_2 + \lambda_{22}t_2^2)\}\phi_{y_1}(t_1 + \beta t_2)\\ &\quad = \exp\{\beta_0^* it_2 - \tfrac{1}{2}(\lambda_{11}^* t_1^2 + 2\lambda_{12}^* t_1t_2 + \lambda_{22}^* t_2^2)\}\phi_{y_1}^*(t_1 + \beta^* t_2).\end{aligned}$$

Suppose next that $\beta \neq \beta^*$. Then we can always find values of t_1 and t_2 such that

$$(3.10)\qquad t_1 + \beta t_2 = z, \qquad t_1 + \beta^* t_2 = 0,$$

where z is arbitrary. We get the solutions

$$\frac{\beta^* z}{\beta - \beta^*}, \qquad t_2 = \frac{z}{\beta - \beta^*}.$$

Substituting these expressions in (3.9), we get

$$\begin{aligned}(3.11)\qquad \phi_{y_1}(z) = \exp\Big\{ i\frac{\beta_0^* - \beta_0}{\beta - \beta^*} z - \frac{z^2}{2(\beta - \beta^*)^2}[(\lambda_{11}^* - \lambda_{11})\beta^{*2}\\ + 2(\lambda_{12}^* - \lambda_{12})\beta^* + \lambda_{22}^* - \lambda_{22}]\Big\};\end{aligned}$$

hence, y_1 is either normally distributed or a constant.[6] In the following a constant will be regarded as a normally distributed variable with variance zero.

If β is finite, it follows that y_2 is also normally distributed. If $\beta = \infty$, equation (3.2) can be written with y_1 expressed in terms of y_2, and we find, in the same way as before, that if β is not identifiable, y_2 must be normally distributed.

Conversely, if both y_1 and y_2 are normally distributed, it is easy to show that β is not identifiable [**12**, Section 1.2]. Hence, we have

THEOREM 1: *A necessary and sufficient condition for β to be identifiable in Model A is that at least one of the variables y_1 and y_2 is not normally distributed.*

We can alternatively express the condition in terms of the joint distribution of the observed variables:

THEOREM 1′: *A necessary and sufficient condition for β to be identifiable in Model A is that the joint distribution of x_1 and x_2 is not normal.*

We shall next consider the identifiability of the rest of the structure

[6] This proof represents a simplification, due to J. Neyman [**17**], of my original proof.

in the case when β is identifiable. For any two equivalent models, (3.9) now holds good with $\beta^* = \beta$.

Let us set, as before, $z = t_1 + \beta t_2$ and substitute in (3.9) $t_1 = z - \beta t_2$. Taking logarithms on both sides, we obtain

$$\begin{aligned}\psi_{v_1}(z) - \psi_v^*(z) &= (\beta_0^* - \beta_0)\, i(z - \beta t_2) + \tfrac{1}{2}(\lambda_{11} - \lambda_{11}^*)(z - \beta t_2)^2 \\ &\quad + (\lambda_{12} - \lambda_{12}^*)(z - \beta t_2)t_2 + \tfrac{1}{2}(\lambda_{22} - \lambda_{22}^*)t_2^2 .\end{aligned} \tag{3.12}$$

Since this is an identity in z and t_2, the coefficients of t_2, zt_2, and t_2^2 must be zero. This gives

$$\beta_0^* = \beta_0 , \tag{3.13}$$

$$\lambda_{12} - \lambda_{12}^* = \beta(\lambda_{11} - \lambda_{11}^*), \tag{3.14}$$

$$\lambda_{22} - \lambda_{22}^* = \beta(\lambda_{12} - \lambda_{12}^*). \tag{3.15}$$

Equation (3.13) shows that β_0 is always identifiable when β is identifiable. Inserting (3.13)–(3.15) in (3.12), we obtain

$$\phi_{v_1}^*(z) = e^{-\frac{1}{2}(\lambda_{11}-\lambda_{11}^*)z^2}\, \phi_{v_1}(z). \tag{3.16}$$

If the three variables x, y, and z are such that $\phi_x(t) = \phi_y(t)\phi_z(t)$, we shall say [13] that the distribution of x is divisible by the distribution of y and divisible by the distribution of z. Now, if the distribution of y_1 is divisible by some normal distribution, the right-hand side of (3.16) must be a characteristic function when $\lambda_{11}^* > \lambda_{11}$ and the difference $\lambda_{11}^* - \lambda_{11}$ is sufficiently small. Determining $\phi_{v_1}^*(z)$, λ_{12}^*, and λ_{22}^* from equations (3.16), (3.14), and (3.15), respectively, we obtain a structure S^* equivalent to S in which the value of β is the same as in S.

If λ_{11} and λ_{22} are both positive, we may choose a value λ_{11}^* such that $\lambda_{11} > \lambda_{11}^* > 0$ and $\lambda_{22} - \beta^2(\lambda_{11} - \lambda_{11}^*) > 0$ and again get a structure S^* equivalent to S such that β has the same value in both structures. Also taking into account the case in which $\beta = \infty$, we may state:

THEOREM 2: *When β is identifiable, a necessary and sufficient condition for the rest of the structure to be identifiable is that both of the following conditions are satisfied: (i) neither the distribution of y_1 nor the distribution of y_2 is divisible by a normal distribution; (ii) either $v_1 = 0$ or $v_2 = 0$.*

4. CONDITIONS FOR THE IDENTIFIABILITY OF A LINEAR RELATION BETWEEN TWO VARIABLES WHEN THE ERRORS ARE INDEPENDENT[7]

In this section we shall consider a model, called Model B, which is defined by specifications (3.1)–(3.3), (3.5), and the following specification:

[7] Part of the results of this section were published previously (*Biometrics*, Vol. 5, March, 1949, pp. 88–89) as an abstract of a paper presented December 29, 1948, at Cleveland, Ohio.

(4.1) v_1 and v_2 are independent.

If $\mathfrak{E}(v_1)$ and $\mathfrak{E}(v_2)$ do not exist, specification (3.5) may be replaced by the specification

(3.5′) median of v_1 = median of $v_2 = 0$.

If x_1 and x_2 are independent, then any value of β is possible, together with the assumption that y_1 and y_2 are constants. In this case, therefore, β is not identifiable.

From now on we shall consider the case where y_1 and y_2 are not independent. Then β is finite and different from zero.

Instead of equation (3.8), we now obtain

$$\phi_{x_1x_2}(t_1, t_2) = \exp\{\beta_0 it_2\}\, \phi_{y_1}(t_1 + \beta t_2)\phi_{v_1}(t_1)\phi_{v_2}(t_2), \tag{4.2}$$

which can also be written

$$\psi_{x_1x_2}(t_1, t_2) = \beta_0 it_2 + \psi_{y_1}(t_1 + \beta t_2) + \psi_{v_1}(t_1) + \psi_{v_2}(t_2). \tag{4.3}$$

It should be noted that the ψ's need not be finite for all values of t_1 and t_2. However, any characteristic function is continuous and equal to one at the origin and must therefore be different from zero in the vicinity of the origin. Hence, the ψ's in equation (4.3) are finite when t_1 and t_2 are sufficiently near to zero.

We shall now define the difference operators

$$\underset{h_1}{\Delta_1}\psi(t_1, t_2) = \psi(t_1 + h_1, t_2) - \psi(t_1, t_2),$$

$$\underset{h_2}{\Delta_2}\psi(t_1, t_2) = \psi(t_1, t_2 + h_2) - \psi(t_1, t_2).$$

We shall apply the difference operator $\underset{h_1}{\Delta_1}\underset{h_2}{\Delta_2}$ to equation (4.3). We then obtain

$$\underset{h_1}{\Delta_1}\underset{h_2}{\Delta_2}\psi_{x_1x_2}(t_1, t_2) = \underset{h_1}{\Delta}\underset{\beta h_2}{\Delta}\psi_{y_1}(t_1 + \beta t_2). \tag{4.4}$$

Suppose now that there exist two different equivalent structures S and S^*, where $\beta \neq \beta^*$. Then we have

$$\underset{h_1}{\Delta}\underset{\beta h_2}{\Delta}\psi_{y_1}(t_1 + \beta t_2) = \underset{h_1}{\Delta}\underset{\beta^* h_2}{\Delta}\psi^*_{y_1}(t_1 + \beta^* t_2). \tag{4.5}$$

Setting t_2 equal to zero and replacing t_1 by $t_1 + \beta^* t_2$, we obtain

$$\underset{h_1}{\Delta}\underset{\beta h_2}{\Delta}\psi_{y_1}(t_1 + \beta^* t_2) = \underset{h_1}{\Delta}\underset{\beta^* h_2}{\Delta}\psi^*_{y_1}(t_1 + \beta^* t_2). \tag{4.6}$$

Combining (4.5) and (4.6) we obtain

$$\underset{h_1}{\Delta}\ \underset{\beta h_2}{\Delta}\ \psi_{y_1}(t_1 + \beta t_2) = \underset{h_1}{\Delta}\ \underset{\beta h_2}{\Delta}\ \psi_{y_1}(t_1 + \beta^* t_2). \tag{4.7}$$

For any z, equations (3.10) will be satisfied by finite values of t_1 and t_2, and when z tends to zero, t_1 and t_2 will also tend to zero. We therefore conclude that the function $\underset{h_1}{\Delta}\ \underset{\beta h_2}{\Delta}\ \psi_{y_1}(z)$ is constant, at least when z, h_1, and h_2 are sufficiently close to zero. But then the function $\psi_{y_1}(z)$ must be a second degree polynomial, at least in the vicinity of the origin; hence y_1 is normally distributed.

As before, we therefore have the result that *β is identifiable if y_1 is not normally distributed.* But the condition will be different from the condition in Model A if we express it in terms of the observed variables. Let κ_{rs} be the semi-invariants of the joint distribution of x_1 and x_2. The condition for the identifiability of β in Model B may now be formulated thus:

THEOREM 3: *If there exists a nonzero (finite or infinite) κ_{rs} with $r \geqslant 1$, $s \geqslant 1$, and either r or s but not both equal to 1, then β is identifiable in Model B.*

When certain semi-invariants exist and are different from zero, β may be expressed in terms of such semi-invariants by one of the equations

$$\kappa_{rs} = \beta\kappa_{r+1,s-1} \qquad (r \geqslant 1,\ s \geqslant 2), \tag{4.8}$$

which are obtained when we put $k = 2$ in equation (2.7).

We shall next examine the identifiability of β in the case where the distribution of y_1 is normal. Let κ_1 and κ_2 be the first two semi-invariants of the distribution of y_1. Equation (4.2) now takes the form

$$\begin{aligned}\phi_{x_1x_2}&(t_1, t_2)\\ &= \exp\left\{i\kappa_1(t_1 + \beta t_2) - \frac{\kappa_2}{2}(t_1 + \beta t_2)^2 + \beta_0 it_2\right\}\phi_{v_1}(t_1)\phi_{v_2}(t_2).\end{aligned} \tag{4.9}$$

Let us again consider two equivalent structures S and S^*. We then obtain from (4.9)

$$\begin{aligned}\exp&\left\{i\kappa_1(t_1 + \beta t_2) - \frac{\kappa_2}{2}(t_1 + \beta t_2)^2 + \beta_0\, it_2\right\}\phi_{v_1}(t_1)\phi_{v_2}(t_2)\\ &= \exp\left\{i\kappa_1^*(t_1 + \beta^* t_2) - \frac{\kappa_2^*}{2}(t_1 + \beta^* t_2)^2 + \beta_0^* it_2\right\}\phi_{v_1}^*(t_1)\phi_{v_2}^*(t_2)\end{aligned} \tag{4.10}$$

Since this is an identity in t_1 and t_2, we immediately obtain

$$\beta\kappa_2 = \beta^*\kappa_2^*. \tag{4.11}$$

Setting $t_2 = 0$ in (4.10), we obtain

$$\phi_{v_1}(t) = \exp\left\{i(\kappa_1^* - \kappa_1)t - \frac{\kappa_2^* - \kappa_2}{2} t^2\right\} \phi_{v_1}^*(t). \tag{4.12}$$

Setting $t_1 = 0$ in (4.10), we obtain

$$\phi_{v_2}(t) = \exp\{i(\kappa_1^*\beta^* - \kappa_1\beta + \beta_0^* - \beta_0)t - \tfrac{1}{2}(\beta^{*2}\kappa_2^* - \beta^2\kappa_2)t^2\}\phi_{v_2}^*(t). \tag{4.13}$$

When $|\beta^*| > |\beta|$, it follows from (4.11) that $\beta^{*2}\kappa_2^* > \beta^2\kappa_2$. When $|\beta^*| < |\beta|$, it follows that $\kappa_2^* > \kappa_2$. In the first case the distribution of v_2, which we shall denote by $P(v_2)$, is divisible by a normal distribution. In the second case $P(v_1)$ is divisible by a normal distribution. Hence, when β is not identifiable, either $P(v_1)$ or $P(v_2)$ is divisible by a normal distribution.

Suppose, conversely, that a structure S is given where the distribution of v_1, say, is divisible by a normal distribution with variance θ. We shall show that there exists an equivalent structure S^* with $\beta^* \neq \beta$. Let us choose κ_2^* such that $\kappa_2 + \theta \geqslant \kappa_2^* > \kappa_2$, and let z be a normally distributed variable with variance $\kappa_2^* - \kappa_2 \leq \theta$. Let

$$\phi_{v_1}^*(t) = \phi_{v_1}(t) \exp\{i(\kappa - \kappa_1^*)t + \tfrac{1}{2}(\kappa_2^* - \kappa_2)t^2\}, \tag{4.14}$$

where κ_1^* is chosen such that specification (3.5) or (3.5′) is satisfied. Let β^* be determined by equation (4.11) and let $\psi_{v_2}^*(t)$ and β_0^* be determined by the equation

$$\phi_{v_2}^*(t) = \phi_{v_2}(t) \exp\{\tfrac{1}{2}(\beta^{*2}\kappa_2^* - \beta^2\kappa_2)t^2 - i(\beta_0^* - \beta_0)t\} \tag{4.15a}$$

such that specification (3.5) or (3.5′) is satisfied. Since (4.11), (4.14), and (4.15a) are satisfied, (4.10) must also hold. Hence, the structure

$$S^* = \{\beta^*, \alpha^*, \kappa_1^*, \kappa_2^*, \psi_{v_1}^*, \psi_{v_2}^*\}$$

is equivalent to S, and $\beta^* \neq \beta$. Hence, β is not identifiable, and we may state:

THEOREM 4: *When y_1 is normally distributed, a necessary and sufficient condition for the identifiability of β is that neither the distribution of v_1 nor the distribution of v_2 is divisible by a normal distribution.*

We shall summarize below in two tables our results concerning the identifiability of β. In Table I we shall give the conditions in terms of the latent variables and parameters. In Table II we shall give the results in terms of the probability distribution of the observed variables.

We shall use the following notations: δ_i is the largest variance of any normal divisor of $P(x_i)$, where $i = 1, 2$; $D = \delta_1\delta_2 - \kappa_{11}^2$.

The statements contained in Table II can be proved as follows. Using

equation (4.3) and the fact that $\psi_{x_1}(t_1) = \psi_{x_1x_2}(t_1, 0)$ and $\psi_{x_2}(t_2) = \psi_{x_1x_2}(0, t_2)$, we obtain

$$\begin{aligned}\psi_{x_1x_2}(t_1, t_2) - \psi_{x_1}(t_1) - \psi_{x_2}(t_2)& \\ = \psi_{y_1}(t_1 + \beta t_2) - \psi_{y_1}(t_1) - \psi_{y_1}(\beta t_2).&\end{aligned} \tag{4.15b}$$

TABLE I

<table>
<tr><th colspan="3">CASE</th><th>CONCLUSION ON β</th></tr>
<tr><td colspan="3">$\beta = 0$ or ∞</td><td>β not identifiable</td></tr>
<tr><td rowspan="3">$\beta \neq 0$ and finite</td><td colspan="2">y_1 not normally distributed</td><td>β identifiable</td></tr>
<tr><td rowspan="2">y_1 normally distributed</td><td>Neither $P(v_1)$ nor $P(v_2)$ divisible by a normal distribution</td><td>β identifiable</td></tr>
<tr><td>Either $P(v_1)$ or $P(v_2)$ divisible by a normal distribution</td><td>β not identifiable</td></tr>
</table>

TABLE II

<table>
<tr><th colspan="2">FORM OF THE FUNCTION
$\psi_{x_1x_2}(t_1, t_2) - \psi_{x_1}(t_1) - \psi_{x_2}(t_2)$</th><th>CONCLUSION ON β</th></tr>
<tr><td colspan="2">Not of the form ct_1t_2</td><td>β identifiable</td></tr>
<tr><td colspan="2">Equal to zero</td><td>β not identifiable</td></tr>
<tr><td rowspan="2">Of the form ct_1t_2, where $c \neq 0$</td><td>$D = 0$</td><td>β identifiable</td></tr>
<tr><td>$D > 0$</td><td>β not identifiable</td></tr>
</table>

When the left-hand side of (4.15b) is identically equal to zero, x_1 and x_2 are independent, and we have seen that β is not identifiable in this case. In the following we shall consider the case where the left-hand side of (4.15b) is not identically equal to zero. When y_1 is normally distributed, $\psi_{y_1}(t)$ is a second-degree polynomial. Substituting a second-degree polynomial in the right-hand side of (4.15b), we see that it is of the form ct_1t_2. Suppose, conversely, that $\psi_{y_1}(t_1 + \beta t_2) - \psi_{y_1}(t_1) - \psi_{y_1}(\beta t_2) = ct_1t_2$. Applying the operator $\underset{h_1}{\Delta_1}$ to this equation we obtain $\underset{h_1}{\Delta}\,\underset{\beta t_2}{\Delta}\,\psi_{y_1}(t) = ch_1t_2$, which shows that $\psi_{y_1}(t)$ is a second-degree polynomial. Hence, y_1

is normally distributed when and only when each side of (4.15b) is of the form ct_1t_2.

Let us now consider the case where y_1 is normally distributed. If v_1 is divisible by some normal distribution, we have $\phi_{x_1}(t) = \phi_{y_1}(t)\phi_{v_1}(t) = \phi_{y_1}(t)\phi_{w_1}(t)\phi_{w_2}(t)$, where $\phi_{w_1}(t)$ is the characteristic function of a normal distribution. It follows that $P(x_1)$ is divisible by a normal distribution whose variance is larger than the variance of y_1.

Suppose, conversely, that $P(x_1)$ is divisible by a normal distribution $P(w_3)$ whose variance is larger than $\text{var}(y_1)$. Then $P(w_3)$ is the product of $P(y_1)$ and another normal distribution $P(w_4)$. It follows that $P(v_1)$ must be divisible by $P(w_4)$, i.e., it is divisible by a normal distribution.

When neither $P(v_1)$ nor $P(v_2)$ is divisible by a normal distribution, it follows that $\delta_1 = \text{var}(y_1)$ and $\delta_2 = \text{var}(y_2)$. Since $\kappa_{11} = \text{cov}(y_1y_2)$ and y_1 and y_2 are linearly dependent, it follows that $D = 0$.

When either $P(v_1)$ or $P(v_2)$ is divisible by a normal distribution, we have either $\delta_1 > \text{var}(y_1)$ or $\delta_2 > \text{var}(y_2)$; hence, $D > 0$. This completes the proof of the statements contained in Table II.

We shall next consider the identifiability of the rest of the structure in the case where β is identifiable. Writing equation (4.3) for two equivalent structures with the same β and subtracting the two equations, we get

$$\begin{aligned}(4.16)\qquad \psi^*_{y_1}(t_1 + \beta t_2) - \psi_{y_1}(t_1 + \beta t_2) + \psi^*_{v_1}(t_1) - \psi_{v_1}(t_1)& \\ + \psi^*_{v_2}(t_2) - \psi_{v_2}(t_2) + (\beta^*_0 - \beta_0)it_2 &= 0.\end{aligned}$$

Applying the operator $\underset{h_1}{\Delta}$ to this equation, we obtain

$$(4.17)\qquad \underset{h_1}{\Delta}\psi^*_{y_1}(t_1 + \beta t_2) - \underset{h_1}{\Delta}\psi_{y_1}(t_1 + \beta t_2) + \underset{h_1}{\Delta}\psi^*_{v_1}(t_1) - \underset{h_1}{\Delta}\psi_{v_1}(t_1) = 0.$$

Putting $t_2 = 0$ in the last equation, thereafter replacing t_1 by $t_1 + \beta t_2$, and subtracting the resulting equation from (4.17), we obtain

$$(4.18)\qquad \underset{h_1}{\Delta}\,\underset{\beta t_2}{\Delta}\,[\psi^*_{v_1}(t_1) - \psi_{v_1}(t_1)] = 0.$$

In the derivation of this equation we have supposed that the functions ψ are finite. This assumption must hold at least in the vicinity of the origin. From (4.18) we conclude that

$$\psi^*_{v_1}(t) - \psi_{v_1}(t) = c_1t + c_0$$

at least in the vicinity of the origin. Since the characteristic function of any distribution is equal to one at the origin, we always have $\psi(0) = 0$. It follows that $c_0 = 0$. Because of specification (3.5) or (3.5′), c_1 must be equal to zero, and we have

$$(4.19)\qquad \psi^*_{v_1}(t) = \psi_{v_1}(t).$$

Combining (4.19) and (4.17), we obtain

$$\psi^*_{y_1}(t) = \psi_{y_1}(t). \tag{4.20}$$

Finally, we obtain from (4.16), (4.19), and (4.20), together with specification (3.5) or (3.5′),

$$\psi^*_{v_2}(t) = \psi_{v_2}(t) \tag{4.21}$$

and

$$\beta^*_0 = \beta_0 . \tag{4.22}$$

Equations (4.19)–(4.21) are not necessarily valid over the whole range of the variable t, but hold at least in the vicinity of the origin. Hence, we conclude that when β is identifiable in Model B, then β_0 is identifiable and the characteristic functions $\phi_{v_1}(t)$, $\phi_{v_2}(t)$, and $\phi_{y_1}(t)$ are identifiable at least in the vicinity of the origin.

If the characteristic functions have discrete zeros only, the characteristic functions will be identifiable over the whole range of the variable t since they are continuous. In this case the whole structure will be identifiable when β is identifiable.

Finally, we shall give an example in which β is identifiable, but where $\phi_{y_1}(t)$ is not identifiable. Let $\phi_1(t)$ and $\phi_2(t)$ be the two characteristic functions given as examples in Cramér's textbook [3, 1945, Section 10.3, p. 94]. Let $\phi_{v_1}(t) = \phi_{v_2}(t) = \phi_1(t)$ and let $\phi_{y_1}(t) = \phi_1[t/(1 + |\beta|)]$. In the structure

$$S = \{\beta, \beta_0, \phi_{y_1}(t), \phi_{u_1}(t), \phi_{u_2}(t)\},$$

β is identifiable since y_1 is not normally distributed. But there exists another equivalent structure,

$$S^* = \{\beta, \beta_0, \phi^*_{y_1}(t), \phi_{v_1}(t), \phi_{v_2}(t)\},$$

where $\phi^*_{y_1}(t) = \phi_2[t/(1 + |\beta|)]$. From the equation

$$\phi_{x_1x_2}(t_1, t_2) = \phi_{y_1}(t_1 + \beta t_2)\phi_{v_1}(t_1)\phi_{v_2}(t_2)e^{i\beta_0 t_2}$$

it follows that $\phi_{x_1x_2}(t_1, t_2) = 0$ in both structures when $|t_1| > 1$ or $|t_2| > 1$. Since $|(t_1 + \beta t_2)/(1 + |\beta|)| < 1$, $\phi_{y_1}(t_1 + \beta t_2)$ and $\phi^*_{y_1}(t_1 + \beta t_2)$ are identical when $t_1 \leqslant 1$ and $t_2 \leqslant 1$. This proves that S and S^* are equivalent.

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NOTICE OF THE CHICAGO MEETING

December 27–30, 1950

The American winter meeting of the Econometric Society will be held in Chicago, Illinois, Wednesday, December 27, to Saturday, December 30, 1950. Sessions of primary interest to economists and those which are sponsored jointly with the American Economic Association and American Farm Economic Association will be held at the Palmer House throughout the period. Sessions which are of interest to statisticians will be held during the first three days of the period at the Congress Hotel, in conjunction with the meetings of the American Statistical Association and Institute of Mathematical Statistics.

Sessions are being planned on the following topics by the program committee of the Econometric Society or co-sponsoring organization: Theory of Comparative Advantage and Patterns of World Trade, Utility Analysis of Decisions Involving Risk, Business Expectations and Business Planning, History of Mathematical Economics, Current Input-Output Studies, Welfare Economics, Problems of Incorrect and Incomplete Specification, Econometric Methods in Agricultural Research, Collection and Use of Survey Data, Demand Analysis, Multivariate Analysis, and Recent Advances in the Theory of Decision Functions. A number of sessions for contributed papers will also be provided.

Further information as to the program and as to hotel reservations will be contained in an announcement to be mailed to American members in November.

CURRENT ACTIVITIES IN ECONOMETRICS

Descriptions of the methods by which its major statistical series are prepared have recently been issued in a concise bulletin by the U. S. Labor Department's Bureau of Labor Statistics. This is the first time that the Bureau has undertaken an over-all job of outlining methodology, although it has been customary to give details on the construction of many of these series in reports dealing with the findings themselves.

In all, thirteen descriptions are included in the bulletin. They cover industrial employment; labor turn-over; earnings and hours in industry; union scales of wages and hours; occupational wages—both the sampling procedures and the conduct of surveys; strike statistics; productivity; industrial injuries; housing volume; expenditures for new construction; monthly and weekly wholesale prices; and the consumers' price index.

To insure maximum usefulness, the individual technical notes were keyed to a relatively standardized outline. In general, the plan followed

has been to give a brief historical statement on each series and to define it, to cite its limitations, to describe the method of collection and the sources, and to explain the actual calculation procedures utilized and the statistical formulas employed.

The thirteen technical notes first appeared in the *Monthly Labor Review* between September, 1949, and April, 1950. They were later brought together in the Bureau's 72-page Bulletin No. 993, "Techniques of Preparing Major BLS Statistical Series." Copies may be obtained at a cost of 40 cents each from the Superintendent of Documents, U. S. Government Printing Office, Washington 25, D. C.

STATISTICS ON MEMBERS AND SUBSCRIBERS

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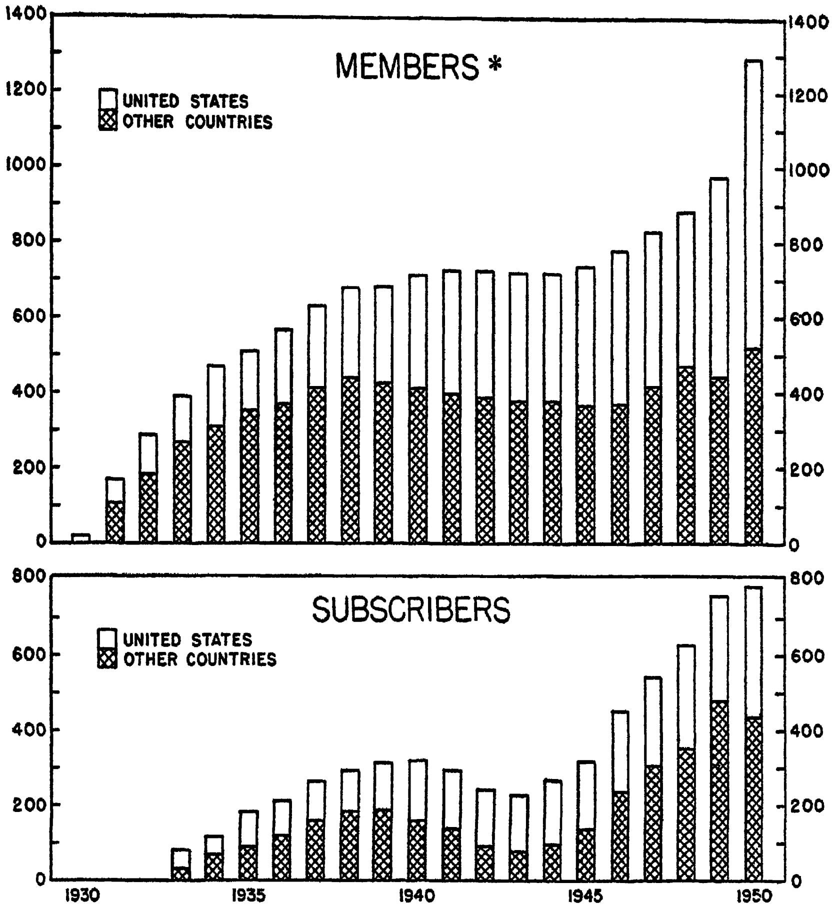
Membership as of June 30, 1949	866	
Members elected, July 1, 1949–September 30, 1950	478	
Members reinstated	5	1,349
Less: Deaths	5	
Resignations	34	
Suspension for nonpayment of dues	16	55
Active membership as of September 30, 1950		1,294

Number of Econometric Society Members per Million of Population, by Countries

Norway	9.59	Austria	.59
Trieste	8.57	Italy	.53
Iceland	8.40	Finland	.51
Curaçao	7.47	Venezuela	.46
United States	5.42	Uruguay	.45
Switzerland	5.16	Spain	.43
Tasmania	4.07	Japan	.41
The Netherlands	3.98	Brazil	.35
Denmark	3.21	Ecuador	.31
Sweden	2.73	Germany	.30
Costa Rica	2.68	Guatemala	.28
Peru	2.53	Portugal	.28
Canada	2.42	Bolivia	.26
New Zealand	1.76	Cuba	.21
France	1.68	Egypt	.17
Great Britain	1.63	Mexico	.15
Panama	1.58	Poland	.15
Belgium	1.55	Bulgaria	.14
Argentina	.99	Colombia	.10
Australia	.95	Burma	.06
Czechoslovakia	.83	Republic of the Philippines	.05
Union of South Africa	.77	Turkey	.05
Chile	.74	India	.04
Ireland	.68	Java	.02
Hungary	.67	Other Countries	.1
Israel	.60		

GROWTH AND DISTRIBUTION OF MEMBERSHIP IN THE ECONOMETRIC SOCIETY AND OF SUBSCRIBERS TO ECONOMETRICA



*Data as of September 30 of each year.

Geographical Distribution of Members and Subscribers

	Members	Subscribers	Total		Members	Subscribers	Total
United States	778	342	1,120	Ireland	2	2	4
Great Britain	70	48	118	Trieste	3	1	4
France	68	21	89	British W. Indies	0	3	3
Italy	24	32	56	Bulgaria	1	2	3
Canada	29	25	54	Costa Rica	2	1	3
The Netherlands	38	15	53	Cuba	1	2	3
Japan	30	16	46	Israel	1	2	3
India	13	31	44	Java	1	2	3
Germany	20	18	38	Pakistan	...	3	3
Norway	30	4	34	Republic of the Philippines	1	2	3
Switzerland	22	12	34	Yugoslavia	...	3	3
Sweden	18	13	31	Burma	1	1	2
Brazil	16	12	28	Ecuador	1	1	2
Belgium	13	10	23	Iceland	1	1	2
Argentina	16	6	22	Peru	1	1	2
Czechoslovakia	10	12	22	Puerto Rico	...	2	2
Spain	11	10	21	Tasmania	1	1	2
Australia	7	12	19	Turkey	1	1	2
Union of South Africa	9	10	19	Uruguay	1	1	2
China	1	16	17	Algeria	...	1	1
Denmark	13	4	17	Bolivia	1	...	1
Poland	5	12	17	Colombia	1	...	1
Hungary	6	10	16	Curaçao	1	...	1
Portugal	2	7	9	Gold Coast	...	1	1
Austria	4	4	8	Guatemala	1	...	1
Chile	4	4	8	Lebanon	...	1	1
New Zealand	3	5	8	Malaya	...	1	1
Mexico	3	4	7	Nigeria	...	1	1
U.S.S.R.	...	7	7	Panama	1	...	1
Ceylon	...	6	6	Rumania	...	1	1
Finland	2	4	6	Salvador	...	1	1
Egypt	3	2	5	Syria	...	1	1
Venezuela	2	3	5				
Greece	...	4	4	Total	1,294	781	2,075

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MISCELLANEOUS